Toroidal Coordinates

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1 Toroidal Coordinates Defined

Problems involving cylindrical symmetry are often most easily solved using cylindrical coordinates. Likewise, problems with spherical symmetry are often most easily solved using spherical coordinates. And in turn, Problems with toroidal symmetry are most easily solved using toroidal coordinates.

Toroidal coordinates is the name I have given to the parametric equations for a torus. Toroidal coordinates can be thought of as a modification of spherical coordinates, so let's begin by first examining spherical coordinates. As the reader is probably familiar with, the transformation from Cartesian to spherical coordinates is given by (see Fig. 1)

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \tag{1}$$

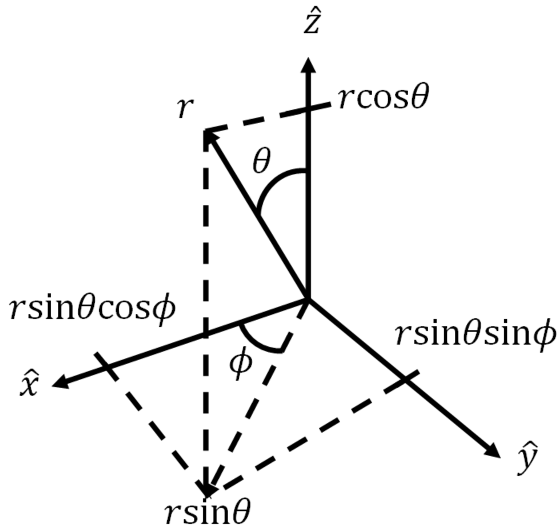

Figure 1: Diagram of spherical coordinates.

This is called spherical coordinates because if r is held constant while θ and ϕ is allowed to vary, the resulting surface is a sphere. How then do we construct a coordinate system that will produce toroidal surfaces? Let's begin by looking at a picture of a torus.

From Fig. 2 we can see that the surface of a torus can be made in a very similar way to how a sphere is made with one modification: the vector **r** needs to be moved from the origin out to the major radius of the torus. Then if we allow θ and ϕ to both vary from 0 to 2π with r constant, we produce the surface of a torus. So how do we codify this mathematically?

With help from Fig. 3, we see that we can use the following transformation:

$$\begin{aligned} x &= (R + r\sin\theta)\cos\phi \\ y &= (R + r\sin\theta)\sin\phi \\ z &= r\cos\theta \end{aligned} \tag{2}$$

The immediate objection to such a transformation is that we have transformed three variables, (x, y, z) into *four* variables (R, r, ϕ, θ). Therefore, every point in space is *not* uniquely defined! But this seeming pitfall can be avoided in many situations of interest and some very elegant solutions can be extracted. First, we

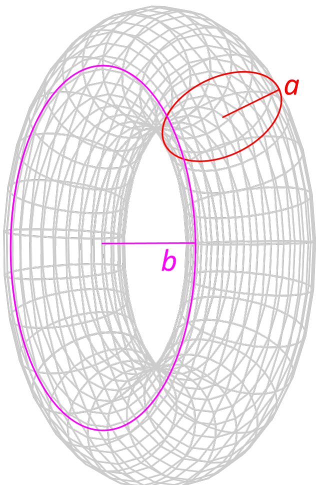

Figure 2: A Torus with the major (b) and minor (a) radius defined

can begin by making R a constant, and choosing the exact size to be based on some defining characteristic of the system in question. Next, we can make the restriction that $r \in [0, R)$. This ensures that everywhere inside the torus is well defined and 1-to-1. This comes at the drawback that there is no way of reaching points outside the torus, but as we shall see in the following examples, this is often of no consequence. In summary, we have for our coordinates $r \in [0, R)$, $\phi \in [0, 2\pi]$, $\theta \in [0, 2\pi]$.

2 Simplified Integrals

What are the principal moments of inertial of a torus? Without toroidal coordinates, this is a very tedious task to calculate involving limits of integration containing square roots of other variables. But, with toroidal coordinates the calculation is simplified dramatically. We begin by computing the Jacobian of equation 2.

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & -(R + r\sin\theta)\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & (R + r\sin\theta)\cos\phi & r\cos\theta\sin\phi \\ \cos\theta & 0 & -r\sin\theta \end{vmatrix} \tag{3}$$

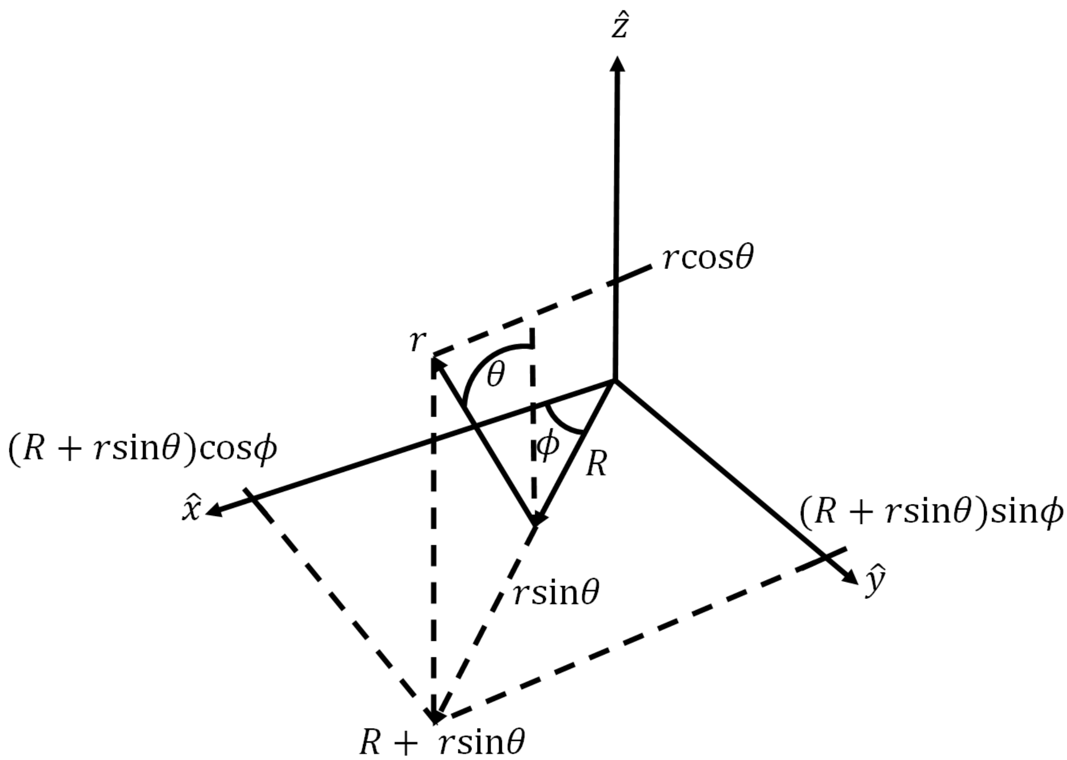

Figure 3: Diagram of toroidal coordinates.

We can use the cofactor expansion method, expanding along the bottom row:

$$
\begin{aligned}
J &= \cos\theta \begin{vmatrix} -(R + r\sin\theta)\sin\phi & r\cos\theta\cos\phi \\ (R + r\sin\theta)\cos\phi & r\cos\theta\sin\phi \end{vmatrix} - r\sin\theta \begin{vmatrix} \sin\theta\cos\phi & -(R + r\sin\theta)\sin\phi \\ \sin\theta\sin\phi & (R + r\sin\theta)\cos\phi \end{vmatrix} \\
&= \begin{aligned} & abs(\cos\theta(-(R + r\sin\theta)r\cos\theta\sin^2\phi - (R + r\sin\theta)r\cos\theta\cos^2\phi) \\ & \quad - r\sin\theta((R + r\sin\theta)\sin\theta\cos^2\phi + (R + r\sin\theta)\sin\theta\cos^2\phi)) \end{aligned} \\
&= \cos\theta((R + r\sin\theta)r\cos\theta) + r\sin\theta((R + \sin\theta)\sin\theta) \\
&= r\cos^2\theta(R + r\sin\theta) + r\sin^2\theta(R + r\sin\theta) \\
&= r(R + r\sin\theta)
\end{aligned}
\tag{4}
$$

As a sanity check, we can compute the volume V of a torus using the fact that $V = \int\int\int dV$. Assuming the Torus has a major radius of b and minor radius of a (as in Fig. 2), the volume is known to be given by

$$V = (\pi a^2)(2\pi b). \tag{5}$$

Meanwhile, using the integral method we have

$$\begin{aligned} V &= \int_0^{2\pi}\int_0^{2\pi}\int_0^a r(b+r\sin\theta)d\phi d\theta dr = 2\pi\int_0^{2\pi}\int_0^a r(b+r\sin\theta)d\theta dr \\ &= 2\pi\int_0^a \left[rb\theta - r\cos\theta\right]_{\theta=0}^{\theta=2\pi} dr = 2\pi b\int_0^a 2\pi r dr = (\pi a^2)(2\pi b) \end{aligned} \tag{6}$$

in agreement with Eq. (5).

As another example, we may use these coordinates to easily compute the principal moments of inertia of a Torus. Refering to the coordinate system shown in Fig. 3, consider a Torus with major radius b and minor radius a, with its axis of symmetry aligned with the z-axis and the xy-plane cutting the Torus into two equal pieces. Then to find the moment of inertial about the x-axis we use the integral

$$I_{xx} = \rho\int_0^{2\pi}\int_0^{2\pi}\int_0^a (y^2+z^2)Jd\phi d\theta dr \tag{7}$$

where I have assumed a constant density ρ. Using Eqs. (2), we can rewrite this integral in terms of the integrating variables and solve. First, expanding the integrand we see

$$\begin{aligned} y^2+z^2 &= (b+r\sin\theta)^2\sin^2\phi + r^2\cos^2\theta \\ &= b^2\sin^2\phi + 2br\sin\theta\sin^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta \end{aligned} \tag{8}$$

and multiplying this by $J = r(b+r\sin\theta) = rb + r^2\sin\theta$ we have

$$\begin{aligned} J(y^2+z^2) =& b^3 r\sin^2\phi + 2b^2r^2\sin^2\phi\sin\theta + br^3\sin^2\phi\sin^2\theta + br^3\cos^2\theta + \\ & b^2r^2\sin^2\phi\sin\theta + 2br^3\sin^2\phi\sin^2\theta + r^4\sin^2\phi\sin^3\theta + r^4\cos^2\theta\sin\theta \end{aligned} \tag{9}$$

Any term with a trig function raised to an odd power will evaluate to zero in the integral since θ and ϕ are both integrated over their full period[1]. To help evaluate the other terms we make use of the fact that $\int_0^{2\pi} cos^2\theta d\theta = \int_0^{2\pi} sin^2\theta d\theta = \pi$. Then, the angular integrals all become very easy and we have

$$\begin{aligned} I_{xx} &= \rho\int_0^a dr(2b^3\pi^2 r + b\pi^2r^3 + 2b\pi^2r^3 + 2b\pi^2r^3) = \rho\int_0^a dr(2b^3\pi^2r + 5b\pi^2r^3) \\ &= \rho(\pi^2b^3a^2 + 5\pi^2ba^4/4). \end{aligned} \tag{10}$$

We can rewrite this in terms of mass using $m = V\rho = 2\pi^2a^2b\rho$ (the final equality follows from Eq. (5)). Then we have

$$I_{xx} = 2\pi^2a^2b\rho(b^2/2 + 5/8a^2) = \frac{1}{8}m(5a^2+4b^2) \tag{11}$$

in agreement with that given in moment of inertia tables.

[1] This is true also for the cubic trig functions because of the triple angle identity: $\sin^3\theta = (3/4)\sin\theta - (1/4)\sin 3\theta$. Clearly, the RHS will evaluate to zero when integrated over a full period.

3 Basis Vectors and Christoffel Symbols (of the Second Kind)

The natural basis vectors of a coordinate system are obtained by differentiating the position vector $\mathbf{x} = x\hat{x} + y\hat{y} + z\hat{z}$ with respect to the new coordinates. That is, $\mathbf{g}_r = \frac{\partial \mathbf{x}}{\partial r}$ etc. Then, by using Eq. (2) we can see that the covariant basis vectors are

$$\mathbf{g}_r = \frac{\partial \mathbf{x}}{\partial r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} \tag{12}$$

$$\mathbf{g}_\theta = \frac{\partial \mathbf{x}}{\partial \theta} = r\cos\theta\cos\phi\hat{x} + r\cos\theta\sin\phi\hat{y} - r\sin\theta\hat{z} \tag{13}$$

$$\mathbf{g}_\phi = \frac{\partial \mathbf{x}}{\partial \phi} = -(R + r\sin\theta)\sin\phi\hat{x} + (R + r\sin\theta)\cos\phi\hat{y} \tag{14}$$

Normalizing these basis vectors results in the same normalized basis vectors found from spherical coordinates. Letting $\mathbf{e}_i = \mathbf{g}_i/\sqrt{\mathbf{g}_i \cdot \mathbf{g}_i}$ we have

$$\mathbf{e}_r = \mathbf{g}_r \tag{15}$$

$$\mathbf{e}_\theta = \mathbf{g}_\theta/r \tag{16}$$

$$\mathbf{e}_\phi = \mathbf{g}_\phi/(R + r\sin\theta) \tag{17}$$

which one can quickly confirm are identical to the unit basis vectors of spherical coordinates. These unit vectors are known to be orthonormal, so solving for the $\mathbf{g}_i$ vectors in terms of the $\mathbf{e}_i$ vectors allows us to quickly compute the covariant metric tensor's components $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$:

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & (R + r\sin\theta)^2 \end{bmatrix} \tag{18}$$

The contravariant metric coefficients are the inverse of this matrix. Since it is diagonal, the inverse is simply found by inverting the diagonal terms leading to

$$[g^{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/(R + r\sin\theta)^2 \end{bmatrix} \tag{19}$$

and this allows us to compute the contravariant natural basis vectors via $\mathbf{g}^j = \mathbf{g}_i g^{ij}$ (sum over i). Hence we have

$$\mathbf{g}^r = \mathbf{g}_r = \mathbf{e}_r \tag{20}$$

$$\mathbf{g}^\theta = \mathbf{g}_\theta/r^2 = \mathbf{e}_\theta/r \tag{21}$$

$$\mathbf{g}^\phi = \mathbf{g}_\phi/(R + r\sin\theta)^2 = \mathbf{e}_\phi/(R + r\sin\theta) \tag{22}$$

And from this we can compute the Christoffel symbols of the second kind Γ^k_{ij} given by

$$\Gamma^k_{ij} = \mathbf{g}^k \cdot \frac{\partial \mathbf{g}_i}{\partial x^j} \tag{23}$$

where x^j is r, θ, ϕ for $i = 1, 2, 3$ respectively. Hence, the non-zero Christoffel symbols of the second kind are

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} \tag{24}$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{\sin\theta}{R + r\sin\theta} \tag{25}$$

$$\Gamma^1_{22} = -r \tag{26}$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \frac{r\cos\theta}{R + r\sin\theta} \tag{27}$$

$$\Gamma^1_{33} = -(R + r\sin\theta)\sin\theta \tag{28}$$

$$\Gamma^2_{33} = (\frac{R}{r} + \sin\theta)\cos\theta. \tag{29}$$

From this we can compute the gradient, divergence, Laplacian, and curl. The gradient of a scalar field s is

$$\begin{aligned} \nabla s &= \frac{\partial s}{\partial r}\mathbf{g}_r + \frac{\partial s}{\partial \theta}\mathbf{g}_\theta + \frac{\partial s}{\partial \phi}\mathbf{g}_\phi \\ &= \frac{\partial s}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial s}{\partial \theta}\mathbf{e}_\theta + \frac{1}{R + r\sin\theta}\frac{\partial s}{\partial \phi}\mathbf{e}_\phi \end{aligned} . \tag{30}$$

The divergence of a vector field $\mathbf{v}$ is

$$\nabla \cdot \mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{R + r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{1}{R + r\sin\theta}(v_r \sin\theta + v_\theta \cos\theta). \tag{31}$$

The Laplacian of a scalar field s is

$$\nabla^2 s = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 s}{\partial \theta^2} + \frac{1}{R + r\sin\theta}\left(\frac{1}{R + r\sin\theta}\frac{\partial^2 s}{\partial \phi} + \frac{\partial s}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial s}{\partial \theta}\cos\theta\right). \tag{32}$$

Finally, the curl of a vector $\mathbf{v}$ is given by

$$\nabla \times \mathbf{v} = \frac{1}{2r(R + r\sin\theta)}\left[\left(\frac{\partial v_\theta}{\partial_\phi} - \frac{\partial v_\phi}{\partial \theta}\right)\mathbf{g}_r + \left(\frac{\partial v_\phi}{\partial r} - \frac{\partial v_r}{\partial \phi}\right)\mathbf{g}_\theta + \left(\frac{\partial v_r}{\partial \theta} - \frac{\partial v_\theta}{\partial r}\right)\mathbf{g}_\phi\right]. \tag{33}$$