Three useful properties of the determinant in index notation

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1 Property I

The determinant of a 3×3 matrix $\mathbf{A}$, $\det(\mathbf{A})$, is given in index notation as

$$\det(\mathbf{A}) = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k}. \tag{1}$$

Because $\det(\mathbf{A}^T) = \det(\mathbf{A})$, we can immediately see that

$$\det(\mathbf{A}) = \det(\mathbf{A}^T) = \varepsilon_{ijk} A_{i1} A_{j2} A_{k3} \tag{2}$$

and therefore

$$\varepsilon_{ijk} A_{1i} A_{2j} A_{3k} = \varepsilon_{ijk} A_{i1} A_{j2} A_{k3}. \tag{3}$$

2 Property II

The determinant can be recast to contain two Levi-Civita symbols

$$\det(\mathbf{A}) = \frac{1}{6} \varepsilon_{lmn} \varepsilon_{ijk} A_{li} A_{mj} A_{nk} \tag{4}$$

which can be see as follows. We begin by examining how Eq. (1) changes if we swap a pair of the fixed indices. For example, if we exchange the indices 1 for 2 and vice versa we arrive at

$$\varepsilon_{ijk} A_{2i} A_{1j} A_{3k}. \tag{5}$$

We can relate this expression back to Eq. (1) by first rearranging,

$$\varepsilon_{ijk} A_{2i} A_{1j} A_{3k} = \varepsilon_{ijk} A_{1j} A_{2i} A_{3k} \tag{6}$$

then swapping i with j and using the fact that $\varepsilon_{jik} = -\varepsilon_{ijk}$

$$\varepsilon_{ijk} A_{1j} A_{2i} A_{3k} = \varepsilon_{jik} A_{1i} A_{2j} A_{3k} = -\varepsilon_{ijk} A_{1i} A_{2j} A_{3k} \tag{7}$$

which thus shows that

$$\varepsilon_{ijk} A_{2i} A_{1j} A_{3k} = -\varepsilon_{ijk} A_{1i} A_{2j} A_{3k}. \tag{8}$$

We can use the same trick to see what happens if we make a cyclic permutation of the fixed indices, i.e. $1 \to 2, 2 \to 3, 3 \to 1$, then we have

$$\varepsilon_{ijk} A_{2i} A_{3j} A_{1k}, \tag{9}$$

and now we rearrange, make the same cyclic permutation on the free indices $i \to j; j \to k; k \to i$, and utilize the fact that $\varepsilon_{ijk} = \varepsilon_{jki}$ to arrive at the result

$$\varepsilon_{ijk} A_{2i} A_{3j} A_{1k} = \varepsilon_{ijk} A_{1k} A_{2i} A_{3j} = \varepsilon_{jki} A_{1i} A_{2j} A_{3k} = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k}. \tag{10}$$

From this we can make the following statement,

$$\varepsilon_{ijk} A_{li} A_{mj} A_{nk} = \begin{cases} \det(\mathbf{A}) & \text{for } \{l, m, n\} = \{1,2,3\}, \{2,3,1\}, \{3,1,2\} \\ -\det(\mathbf{A}) & \text{for } \{l, m, n\} = \{1,3,2\}, \{2,1,3\}, \{3,2,1\} \\ \text{something else} & \text{for } \{l, m, n\} = \text{all else.} \end{cases} \tag{11}$$

The existence of the determinant is enough to guarantee that the final line is non-singular. We can thus use Eq. (11) to make the following evaluation:

$$\varepsilon_{lmn} \varepsilon_{ijk} A_{li} A_{mj} A_{nk} = 6 \det(\mathbf{A}) \tag{12}$$

or equivalently,

$$\det(\mathbf{A}) = \frac{1}{6} \varepsilon_{lmn} \varepsilon_{ijk} A_{li} A_{mj} A_{nk} \tag{13}$$

as was desired.

3 Property III

We can use (11) to extract another useful equation. Since the final line will result in a number of no meaningful value, rather than placing a second Levi-Civita symbol on the left-hand side of the equation as we did before, we may place it also on the right hand side to dispense with these nonsense numbers and arrive at

$$\varepsilon_{ijk} A_{li} A_{mj} A_{nk} = \varepsilon_{lmn} \det(\mathbf{A}). \tag{14}$$

This result is useful in that it effectively sums over the dummy indicies $\{i, j, k\}$ leaving only the free indices remaining.