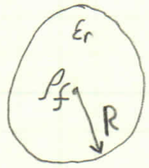
4.20




Exploiting symmetry and using the Gauss'-like law,

$$DA = \rho_f V \Rightarrow D = \rho_f \frac{V}{A}, \quad \frac{V}{A} = \rho_f \frac{\frac{4}{3}\pi R^3}{4\pi r^2} = \frac{\rho_f}{3}\frac{R^3}{r^2}, \quad r > R$$

$$\Rightarrow \vec{D} = \rho_f \frac{R^3}{3}\frac{\hat{r}}{r^2}$$ and for a linear dielectric

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} \Rightarrow \vec{E} = \rho_f \frac{R^3}{\varepsilon_0 \varepsilon_r 3}\frac{\hat{r}}{r^2}$$

$$V(\vec{r}\,|\,r>R) = -\int \vec{E}\cdot d\vec{l} = -\int_\infty^r \rho_f \frac{R^3}{3\varepsilon_0\varepsilon_r}\frac{dr}{r^2} = \rho_f \frac{R^3}{3\varepsilon_0\varepsilon_r}\left[\frac{1}{r}\right]_\infty^r = \rho_f \frac{R^3}{3\varepsilon_0\varepsilon_r}\frac{1}{r}$$

For $r<R$,

$$D = \rho_f \frac{V}{A} = \rho_f \frac{\frac{4}{3}\pi r^3}{4\pi r^2} = \rho_f \frac{r}{3} \Rightarrow \vec{D} = \frac{r\hat{r}}{3} = \varepsilon_0 \varepsilon_r \vec{E} \Rightarrow \vec{E} = \frac{r\hat{r}}{3\varepsilon_0\varepsilon_r}$$

and so

$$V(\vec{r}\,|\,r<R) = -\int E\cdot dl = -\int_\infty^R \frac{R^3}{3\varepsilon_0\varepsilon_r}\frac{dr}{r^2} - \int_R^r \frac{r\,dr}{3\varepsilon_0\varepsilon_r} = \rho_f \frac{R^2}{3\varepsilon_0\varepsilon_r} - \rho_f \frac{1}{6\varepsilon_0\varepsilon_r}\left[r^2 - R^2\right]$$

but in the second term, $\varepsilon_r$ is different from the 1st term, since in the 1st term, we are in vacuum, so $\varepsilon_r = 1$ so,

$$\rho_f \frac{2\varepsilon_r R^2}{6\varepsilon_0\varepsilon_r} + \rho_f \frac{r^2 - R^2}{6\varepsilon_0\varepsilon_r} = \rho_f \frac{(2\varepsilon_r + 1)R^2 - r^2}{6\varepsilon_0\varepsilon_r}$$

so

$$V(\vec{r}) = \begin{cases} \rho_f \dfrac{R^3}{3\varepsilon_0}\dfrac{1}{r} & \text{for } r > R \\ \rho_f \dfrac{(2\varepsilon_r+1)R^2 - r^2}{6\varepsilon_0\varepsilon_r} & \text{for } r < R \end{cases}$$

and so $V(\vec{0}) = \rho_f \dfrac{(2\varepsilon_r + 1)}{6\varepsilon_0\varepsilon_r}R^2 = \dfrac{\rho_f R^2}{3\varepsilon_0}\left(\dfrac{2\varepsilon_r}{2\varepsilon_r} + \dfrac{1}{2\varepsilon_r}\right) = \dfrac{\rho_f R^2}{3\varepsilon_0}\left(1 + \dfrac{1}{2\varepsilon_r}\right)$