Hinkle, Zachary | EM Griffiths | 08/13/18 | 1/2

4.2 Here $\rho(\vec{r}) = \frac{q}{\pi a^3} e^{-2r/a}$. Since ρ is radially symmetric, we can exploit Gauss's law to find $\vec{E}$.

$$\oint \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0}\int_V \rho(\vec{r})\, d\tau$$

$$\oint \vec{E}\cdot d\vec{a} = E\oint da = 4\pi r^2 E.$$

$$\frac{1}{\varepsilon_0}\int_V \rho(\vec{r})\, d\tau = \frac{4\pi q}{\pi a^3 \varepsilon_0}\int r^2 e^{-2r/a}\, dr$$

$f''g + 2g'f' + fg''$

$f''g + f''g' + 2g''f' + 2g f''' + f'g'' + fg'''$

Let $u = r^2 \Rightarrow du = 2r\,dr$
$dv = e^{-2r/a} dr \Rightarrow v = -\frac{a}{2} e^{-2r/a}$, then

$$\int r^2 e^{-2r/a} dr = -\frac{ar^2}{2} e^{-2r/a} + a\int r e^{-2r/a} dr.$$

Let $u' = ar \Rightarrow du' = a\,dr$
$dv' = e^{-2r/a} dr \Rightarrow v = -\frac{a}{2} e^{-2r/a}$

$$a\int r e^{-2r/a} dr = -\frac{a^2 r}{2} e^{-2r/a} + \frac{a^2}{2}\int e^{-2r/a} dr = -\frac{a^2 r}{2} e^{-2r/a} - \frac{a^3}{4} e^{-2r/a}$$

So

$$\frac{1}{\varepsilon_0}\int_V \rho(\vec{r})\, d\tau = -\frac{4q}{a^3\varepsilon_0} e^{-2r/a}\left[\frac{a}{2}r^2 + \frac{a^2 r}{2} + \frac{a^3}{4}\right]_0^r = \frac{4q}{\varepsilon_0} e^{-2r/a}\left[\frac{1}{2}\left(\frac{r}{a}\right)^2 + \frac{1}{2}\left(\frac{r}{a}\right) + \frac{1}{4}\right]_r^0$$

$$= \frac{4q}{\varepsilon_0}\left[\frac{1}{4} - e^{-2r/a}\left\{\frac{1}{2}\left(\frac{r}{a}\right)^2 + \frac{1}{2}\left(\frac{r}{a}\right) + \frac{1}{4}\right\}\right] \Rightarrow \vec{E} = \frac{q}{\pi\varepsilon_0}\left[\frac{1}{4} - e^{-2r/a}\left\{\frac{1}{2}\left(\frac{r}{a}\right)^2 + \frac{1}{2}\left(\frac{r}{a}\right) + \frac{1}{4}\right\}\right]$$

Then we can expand in a Taylor Series

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \frac{1}{3!}f'''(x_0)(x-x_0)^3 + \ldots$$

$$(r^2E)'(0) = \frac{q}{\pi\varepsilon_0}\left[\frac{2}{a}\left(\frac{1}{4}\right) - \frac{1}{2a}\right] = 0$$

$$\frac{\pi\varepsilon_0}{q}(r^2E)''(0) = -\frac{4}{a^2}\left(\frac{1}{4}\right) + 2\left(\frac{2}{a}\right)\left(\frac{1}{2a}\right) + (-1)\left(\frac{1}{a^2}\right) = 0$$

$$\frac{\pi\varepsilon_0}{q}(r^2E)'''(0) = \frac{8}{a^3}\left(\frac{1}{4}\right) + 3\left(-\frac{4}{a^2}\right)\left(\frac{1}{2a}\right) + 3\left(\frac{2}{a}\right)\left(\frac{1}{a^2}\right) + (-1)(0)$$

$$= \frac{2}{a^3} - \frac{6}{a^3} + \frac{63}{a^3} = \frac{2}{a^3}$$

$$\Rightarrow r^2E(\vec{r}) \approx \frac{1}{3!}\left(\frac{2}{a^3}\right)r^3\left(\frac{q}{\pi\varepsilon_0}\right)\hat{r} = \frac{q}{3\pi\varepsilon_0}\left(\frac{r}{a}\right)^3\hat{r}$$

$$\Rightarrow \vec{E}(\vec{r}) \approx \frac{qr}{3\pi\varepsilon_0 a^3}\hat{r}$$

$\vec{p} = \alpha\vec{E}$. w/ $\hat{p} = \hat{E}$,

$$qd = \alpha\frac{qd}{3\pi\varepsilon_0 a^3} \Rightarrow \alpha = 3\pi\varepsilon_0 a^3$$