3.19 As always, I look first for a Legendre Expansion of the potential.

Here, $V(\theta) = \cos(3\theta) = \left(4\cos^3\theta - 3\cos\theta\right)k$

$P_3(x) = \frac{1}{2}(5x^3 - 3x) \Rightarrow \frac{8}{5}P_3(x) = 4x^3 - \frac{12}{5}x$

$P_1(x) = x \Rightarrow \frac{8}{5}P_3(x) - \frac{3}{5}P_1(x) = 4x^3 - 3x$

so $V(\theta) = \left(\frac{8}{5}P_3(\cos\theta) - \frac{3}{5}P_1(\cos\theta)\right)k$

then Inside

$$A_l = \frac{2l+1}{2R^l}\int_0^\pi V_0(\theta)P_l(\cos\theta)\sin\theta\, d\theta$$

$$A_1 = k\frac{3}{2R}\left(\frac{-3}{5}\right)\int_0^\pi P_1^2(\cos\theta)\sin\theta\, d\theta = k\frac{3}{2R}\left(\frac{-3}{5}\right)\left(\frac{2}{3}\right) = \frac{-3k}{5R}$$

$$A_3 = k\frac{7}{2R^3}\left(\frac{8}{5}\right)\int_0^\pi P_3^2(\cos\theta)\sin\theta\, d\theta = k\frac{7}{2R^3}\left(\frac{8}{5}\right)\left(\frac{2}{7}\right) = \frac{4k}{5R^3}$$

all other $A_l = 0$.

$$V(r,\theta) = \left(-\frac{3}{5}k\right)\left(\frac{r}{R}\right)P_1(\cos\theta) + \left(\frac{4k}{5}\right)\left(\frac{r}{R}\right)^3 P_3(\cos\theta)$$

Outside

$$B_l = \frac{2l+1}{2}R^{l+1}\int_0^\pi V_0 P_l(\cos\theta)\sin\theta\, d\theta$$

$$B_1 = k\left(\frac{2l+1}{2}\right)R^2\left(\frac{-3}{5}\right)\int_0^\pi P_1^2(\cos\theta)\sin\theta\, d\theta = \left(\frac{2l+1}{2}\right)\left(\frac{2}{2l+1}\right)\left(-\frac{3}{5}\right)kR^2 = -\frac{3}{5}kR^2$$

$$B_3 = k\left(\frac{2l+1}{2}\right)R^4\left(\frac{8}{5}\right)\int_0^\pi P_3^2(\cos\theta)\sin\theta\, d\theta = \left(\frac{2l+1}{2}\right)\left(\frac{2}{2l+1}\right)R^4 k\left(\frac{8}{5}\right) = \frac{8}{5}kR^4$$

$$V(r,\theta) = -\frac{3}{5}k\left(\frac{R}{r}\right)^2 P_1(\cos\theta) + \frac{8}{5}k\left(\frac{R}{r}\right)^4 P_3(\cos\theta)$$

b)

$$\sigma_0(\theta) = \varepsilon_0\left(\frac{\partial V_{in}}{\partial r} - \frac{\partial V_{out}}{\partial r}\right)$$

$$\frac{\partial V_{in}}{\partial r} = \left(\frac{-3}{5R}\right) k\, P_1(\cos\theta) + \left(\frac{12k}{5R}\right)\left(\frac{r}{R}\right)^2 P_3(\cos\theta)$$

$$\frac{\partial V_{out}}{\partial r} = \frac{6k}{5r}\left(\frac{R}{r}\right)^2 P_1(\cos\theta) - \frac{32k}{5r}\left(\frac{R}{r}\right)^4 P_3(\cos\theta)$$

$$\frac{\partial V_{in}}{\partial r} - \frac{\partial V_{out}}{\partial r} = -k\left(\frac{3}{5R} + \frac{6R^2}{5r^3}\right) P_1(\cos\theta) + k\left(\frac{12r^2}{5R^3} + \frac{32R^4}{5r^5}\right) P_3(\cos\theta)$$

$$= -k\left(\frac{3R^3 + 6r^3}{5(Rr)^3}\right) P_1(\cos\theta) + k\left(\frac{12r^7 + 32R^7}{5R^3r^5}\right) P_3(\cos\theta)$$

$$\sigma_0(\theta) = k\varepsilon_0\left[\left(\frac{12r^7 + 32R^7}{5R^3r^5}\right) P_3(\cos\theta) - 3\left(\frac{2r^3 + R^3}{5R^3r^2}\right) P_1(\cos\theta)\right]$$