3.18 a) Since the Legendre polynomials form a complete, orthogonal basis,

The trick here is to first write V_0 as a Legendre series.

That is $V_0(\theta) = \sum_{\ell=1}^{\infty} C_\ell P_\ell(\cos\theta)$. To determine each C_ℓ, multiply by $P_{\ell'}(\cos\theta)\sin\theta\,d\theta$ and integrate.

Then $\int V_0(\theta) P_{\ell'}(\cos\theta)\sin\theta\,d\theta = \left(\frac{2}{2\ell'+1}\right) C_{\ell'}$

So
$$C_\ell = \left(\frac{2\ell+1}{2}\right)\int V_0(\theta) P_\ell(\cos\theta)\sin\theta\,d\theta$$

For $V_0(\theta) = V_0$ a constant,
$$C_\ell = \left(\frac{2\ell+1}{2}\right) V_0 \int_{-1}^{1} P_\ell(x)\,dx = \begin{cases} V_0 & \text{for } \ell = 0 \\ 0 & \text{for } \ell > 0 \end{cases}$$

So $V_0(\theta) = V_0 P_0(\cos\theta)$. This is probably pretty obvious from Table 3.1, but I thought the Legendre series was instructive.

Anyway,
$$A_\ell = \frac{2\ell+1}{2R^\ell}\int_0^\pi V_0 P_0(\cos\theta) P_\ell(\cos\theta)\sin\theta\,d\theta = \begin{cases} V_0 & \text{for } \ell = 0 \\ 0 & \text{for } \ell > 0 \end{cases}$$

So inside the sphere,
$$V(r,\theta) = V_0 \quad (r \in [0,R]).$$

Outside the sphere,
$$B_\ell = \frac{2\ell+1}{2} R^{\ell+1} V_0 \int_0^\pi P_0(\cos\theta) P_\ell(\cos\theta)\sin\theta\,d\theta = \begin{cases} V_0 R^1 & \text{for } \ell = 0 \\ 0 & \text{for } \ell > 0 \end{cases}$$

and hence
$$V(r,\theta) = V_0\left(\frac{R}{r}\right) \quad (r \in (R,\infty))$$

If $\sigma(\theta)$ is constant then $\sigma(\theta) = \sigma_0 P_0(\cos\theta)$

So

$$A_\ell = \frac{1}{2\varepsilon_0 R^{\ell-1}} \int_0^\pi \sigma_0 P_0(\cos\theta) P_\ell(\cos\theta) \sin\theta d\theta \Rightarrow A_0 = \frac{\sigma_0}{2\varepsilon_0 R^{-1}} \left(\frac{2}{0+1}\right) = \frac{R}{\varepsilon_0};$$

So $B_\ell = \dfrac{R^{2(\ell+1)}}{\varepsilon_0} \Rightarrow B_0 = \sigma_0 \dfrac{R^2}{\varepsilon_0}$ and

$$V(r,\theta) = \begin{cases} \sigma_0 \frac{R}{\varepsilon_0} & \text{for } r < R \\ \sigma_0 \frac{R}{\varepsilon_0}\left(\frac{R}{r}\right) & \text{for } r > R \end{cases}$$