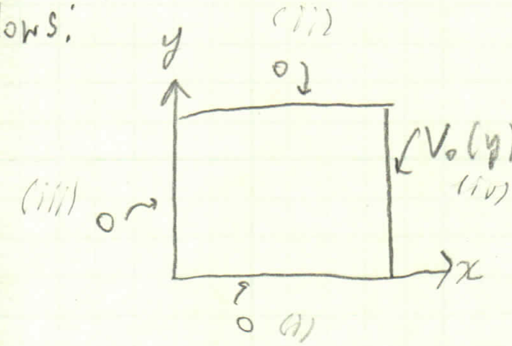
3.15 The boundary conditions are as follows:

(i) $V(x,0)=0$

(ii) $V(x,a)=0$

(iii) $V(0,y)=0$

(iv) $V(a,y)=V_0(y)$



a) inside the pipe $\{V\in\mathbb{R}^3 \mid x,y\in(0,a),\ z\in(-\infty,\infty)\}$

and $\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}=0 \qquad (1)$

Let $V(x,y)=X(x)\,Y(y)$ so that (1) yields

$\dfrac{1}{X}\dfrac{d^2X}{dx^2}=C_1$ & $\dfrac{1}{Y}\dfrac{d^2Y}{dy^2}=C_2$ and $C_1+C_2=0$.

let $C_1=k^2$

then

$V(x,y)=(Ae^{kx}+Be^{-kx})(C\sin ky+D\cos ky)$.

from (i) $(Ae^{kx}+Be^{-kx})(D)=0 \Rightarrow D=0$

from (ii) $(Ae^{kx}+Be^{-kx})(C\sin ka)=0 \Rightarrow k=\frac{n\pi}{a}$

from (iii) $(A+B)C\sin(ky)=0 \Rightarrow A=-B \Rightarrow X(x)=2\sinh(kx)$

So

$V_n(x,y)=C_n\sinh(kx)\sin(ky)$ and $V(x,y)=\sum_{n=1}^{\infty}C_n\sinh(kx)\sin(ky)$

and we require from (iv) that

$V(a,y)=\sum V_n(a,y)=V_0(y)$

since the basis $\sin\left(\frac{n\pi y}{a}\right)$ is orthogonal,

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)dy = \begin{cases} 0 & n \neq n' \\ \frac{a}{2} & n = n' \end{cases}$$ so for any particular n,

$$\int \sum C_n \sinh(ka)\sin(ky)\sin(k'y)\,dy) = \int V_0(y)\sin(k'y)\,dy$$

$$\Rightarrow \frac{a}{2}\sinh(ka)\,C_{n'} = \int_0^a V_0(y)\sin(k'y)\,dy$$

$$\Rightarrow C_n = \frac{2}{a\sinh(ka)}\int_0^a V_0(y)\sin(k'y)\,dy$$

So the general form solution is

$$V(x,y) = \frac{2}{a\sinh(ka)}\sum_{n=1}^{\infty}\left[\sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy\right]$$

b) for $V_0(y) = V_0$ a constant,

let $u = \frac{n\pi y}{a} \Rightarrow du = \frac{n\pi}{a}dy \Rightarrow dy = \frac{a}{n\pi}du$

$$\frac{a}{2}\sinh(n\pi)\,C_n = V_0\int_0^a \sin\left(\frac{n\pi y}{a}\right)dy = \frac{a}{n\pi}\int_0^{n\pi}\sin u\,du = \frac{a}{n\pi}(\cos u)_{n\pi}^{0}$$

$$\Rightarrow C_n = \begin{cases} 0 \text{ for } n \in 2\mathbb{N} \\ \frac{4V_0}{n\pi\sinh(n\pi)} \text{ for } n \in 2\mathbb{N}-1 \end{cases}$$

So

$$V(x,y) = \frac{4V_0}{n\pi}\sum_{n\in 2\mathbb{N}-1}^{\infty}\frac{\sinh\left(\frac{n\pi x}{a}\right)}{n\sinh(n\pi)}\sin\left(\frac{n\pi y}{a}\right)$$