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3.3 If $V(r)$ then $\nabla^2 V(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = \frac{2r}{r^2}\frac{\partial V}{\partial r} + \frac{r^2}{r^2}\frac{\partial^2 V}{\partial r^2}$

and so Laplace's equation $\nabla^2 V = 0$, becomes

$\frac{2}{r}\frac{dV}{dr} + \frac{d^2 V}{dr^2} = 0$. Let $u(r) = \frac{dV}{dr}$

so

$u' + \frac{2}{r}u = 0$. Use the integrating factor method.

$u = C_1 e^{-\int \frac{2}{r} dr}$

$2\int \frac{dr}{r} = 2\ln|r|$ but since $r \geq 0$, $|r| = r$

so

$u = \frac{C_1}{r^2}$ $\quad \frac{dV}{dr} = \frac{C_1}{r^2} \Rightarrow V = \frac{-C_1}{r} + C_2$

as a check $\frac{dV}{dr} = \frac{C_1}{r^2}$ $\quad \frac{d^2V}{dr^2} = \frac{-2C_1}{r^3}$

$\frac{2}{r}\frac{dV}{dr} = \frac{2C_1}{r^3}$ So $\frac{2}{r}\frac{dV}{dr} + \frac{d^2V}{dr^2} = \frac{2C_1}{r^3} - \frac{2C_1}{r^3} = 0$

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If $V(s)$ then $\nabla^2 V(s) = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) = \frac{1}{s}\frac{\partial V}{\partial s} + \frac{s}{s}\frac{\partial^2 V}{\partial s^2}$

so $\frac{1}{s}\frac{dV}{ds} + \frac{d^2V}{ds^2} = 0$ letting $u = \frac{dV}{ds}$

$u' + \frac{1}{s}u = 0$ so $u = C_1 e^{-\int \frac{ds}{s}} = \frac{C_1}{s}$

so $\frac{dV}{ds} = \frac{C_1}{s} \Rightarrow V = C_1 \ln s + C_2$

As a check $\frac{d^2V}{ds^2} = \frac{d}{ds}\left(\frac{C_1}{s}\right) = \frac{-C_1}{s^2}$ ; $\frac{1}{s}\frac{dV}{ds} = \frac{C_1}{s^2}$ so

$\frac{1}{s}\frac{dV}{ds} + \frac{d^2V}{ds^2} = \frac{C_1}{s^2} - \frac{C_1}{s^2} = 0$ as expected.