From pg. 142 | [illegible], Zachary | EM Griffiths | 07/14/18 | 1/1

Note:

Eq. 58 is in the form of an Euler-Cauchy equation which has the following general form:

$$y'' + \frac{\alpha}{t} y' + \beta y = 0$$

here, $\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = \ell(\ell+1) R$ and $\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = 2r\frac{dR}{dr} + r^2 \frac{d^2R}{dr^2}$

So $\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} - \frac{\ell(\ell+1)}{r^2}R = 0$, which matches the general form above.

The characteristic eq. to an Euler-Cauchy equation is

$\lambda^2 + (\alpha - 1)\lambda + \beta = 0$ with general solution

$y(t) = C_1 t^{\lambda_1} + C_2 t^{\lambda_2}$

here, $\lambda^2 + \lambda - \ell(\ell+1) = 0 \Rightarrow \lambda = \frac{-1 \pm \sqrt{1 + 4\ell(\ell+1)}}{2} = \frac{-1 \pm \sqrt{4\ell^2 + 4\ell + 1}}{2}$

$$= \frac{-1 \pm \sqrt{(2\ell+1)^2}}{2} = \frac{-1 \pm (2\ell+1)}{2}$$

$$= \frac{-1 + 2\ell + 1}{2}, \frac{-1 - 2\ell - 1}{2} = \frac{2\ell}{2}, \frac{-2-2\ell}{2} = \ell, -(\ell+1)$$

So $R(r) = C_1 r^\ell + \frac{C_2}{r^{\ell+1}}$ as Griffiths claims.

See Zachary S. Tseng for a proof of the Euler-Cauchy solution.