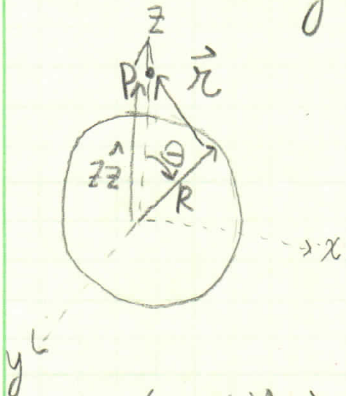
2.7



$$z\hat{z} = R\hat{r} + \vec{\mathscr{r}} \Rightarrow \vec{\mathscr{r}} = z\hat{z} - R\hat{r}$$

$$\vec{\mathscr{r}} = z\hat{z} - R\sin\theta\cos\phi\,\hat{x} - R\sin\theta\sin\phi\,\hat{y} - R\cos\theta\,\hat{z}$$

$$\mathscr{r}^2 = (z - R\cos\theta)^2 + R^2\sin^2\theta\,(\cos^2\phi + \sin^2\phi)$$

$$\mathscr{r}^2 = z^2 - 2zR\cos\theta + R^2\cos^2\theta + R^2\sin^2\theta$$

$$E = k\int \frac{\sigma(\vec{r}\,')\,da}{\mathscr{r}^3}\vec{\mathscr{r}} \qquad \mathscr{r}^2 = z^2 - 2zR\cos\theta + R^2$$

$$\mathscr{r} = \sqrt{z^2 + R^2 - 2zR\cos\theta}$$

$$\frac{E}{k\rho} = \int \frac{(z - R\cos\theta)\hat{z} - R\sin\theta\cos\phi\,\hat{x} - R\sin\theta\sin\phi\,\hat{y}}{(z^2 + R^2 - 2zR\cos\theta)^{3/2}} R^2\sin\theta\, d\theta\, d\phi$$

the x & y components are zero since $\phi \in [0, 2\pi)$

$$\frac{E}{2\pi R^2 k\sigma} = \int \frac{(z - R\cos\theta)\hat{z}\sin\theta}{(z^2 + R^2 - 2zR\cos\theta)^{3/2}} d\theta \qquad \text{let } w = z^2 + R^2 - 2zR\cos\theta$$

$$dw = +2zR\sin\theta\, d\theta$$

$$-\int \frac{R\sin\theta\, d\theta}{(z^2 + R^2 - 2zR\cos\theta)^{3/2}} = -\frac{1}{2z}\int w^{-3/2}\,dw = \frac{1}{z} w^{-1/2} = \frac{1}{z\sqrt{z^2 + R^2 - 2zR\cos\theta}}$$

let $u = \cos\theta \Rightarrow du = -\sin\theta\, d\theta$

and $dv = -\dfrac{R\sin\theta\, d\theta}{(z^2 + R^2 - 2zR\cos\theta)^{3/2}} \Rightarrow v = \dfrac{1}{z\sqrt{z^2 + R^2 - 2zR\cos\theta}}$

then

$$I_1 = -\int \frac{R\cos\theta\sin\theta\, d\theta}{(z^2 + R^2 - 2zR\cos\theta)^{3/2}} = \frac{\cos\theta}{z\sqrt{z^2 + R^2 - 2zR\cos\theta}} + \int \frac{\sin\theta\, d\theta}{z\sqrt{z^2 + R^2 - 2zR\cos\theta}}$$

using w as before

$$\int \frac{\sin\theta\, d\theta}{z\sqrt{z^2 + R^2 - 2zR\cos\theta}} = \frac{1}{2z^2R}\int w^{-1/2}\,dw = \frac{1}{z^2R}\sqrt{z^2 + R^2 - 2zR\cos\theta}$$

2.7 So

$$I_1 = \left[\frac{\cos\theta}{z\sqrt{z^2+R^2-2zR\cos\theta}} + \frac{1}{z^2R}\sqrt{z^2+R^2-2zR\cos\theta} \right]_0^{\pi}$$

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for $z > R$

$$I_1 = \frac{-1}{z(z+R)} + \frac{-1}{z(z-R)} + \frac{(z+R)-(z-R)}{z^2R} = \frac{2}{z^2} - \frac{z-R+z+R}{z(z^2-R^2)}$$

$$= \frac{2}{z^2} - \frac{2}{(z^2-R^2)} = \frac{2z^2-2R^2-2z^2}{z^2(z^2-R^2)} = \frac{-2R^2}{(z^2-R^2)z^2}$$

$$I_2 = \int \frac{z\sin\theta\, d\theta}{(z^2+R^2-2zR\cos\theta)^{3/2}} = \left[\frac{-1}{R\sqrt{z^2+R^2-2zR\cos\theta}} \right]_{\pi}^{0}$$

for $z > R$

$$= \frac{1}{R}\left(\frac{1}{z-R} - \frac{1}{R+z} \right) = \frac{1}{R}\left(\frac{R+z-z+R}{z^2-R^2} \right) = \frac{2}{z^2-R^2} = \frac{2z^2}{z^2(z^2-R^2)}$$

$$\frac{E}{2\pi R^2 k\sigma}\cdot\hat{z} = I_1 + I_2 = \frac{2z^2-2R^2}{z^2(z^2-R^2)} = \frac{2}{z^2}$$

$$\Rightarrow E = \frac{1}{4\pi\varepsilon_0}\cdot\frac{4\pi R^2\sigma}{z^2}\hat{z} = \frac{q}{4\pi\varepsilon_0 z^2}\hat{z}$$ for $z > R$ since $4\pi R^2\sigma$ is the total charge on the surface

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for $z < R$

$$I_1 = \frac{-1}{z}\left(\frac{1}{z+R} + \frac{1}{R-z} \right) + \frac{(z+R)-(R-z)}{z^2R} = \frac{2}{zR} - \frac{R-z+z+R}{z(R^2-z^2)}$$

$$= \frac{2}{zR} - \frac{2R}{z(R^2-z^2)} = \frac{2(R^2-z^2)-2R^2}{zR(R^2-z^2)} = \frac{-2z^2}{zR(R^2-z^2)}$$

$$I_2 = \frac{1}{R}\left(\frac{1}{R-z} - \frac{1}{R+z} \right) = \frac{1}{R}\left(\frac{R+z-R+z}{R^2-z^2} \right) = \frac{2z}{R(R^2-z^2)} = \frac{2z^2}{zR(R^2-z^2)}$$

Then $I_1 + I_2 = \dfrac{-2z^2}{zR(R^2-z^2)} + \dfrac{2z^2}{zR(R^2-z^2)} = 0$

So

$\vec{E} = 0$ for $z<R$

hence

$$\vec{E} = \begin{cases} \dfrac{q}{4\pi\varepsilon_0 z^2}\hat{z} & \text{for } z>R \\ 0 & \text{for } z<R \end{cases}$$

Since the sphere is radially symetric about θ & ϕ, our choice of the z-axis is entirely arbitrary so we can simply rewrite the field as

$$\vec{E} = \begin{cases} \dfrac{q}{4\pi\varepsilon_0 r^3}\vec{r} & \text{for } r>R \\ 0 & \text{for } r<R \end{cases}$$

which means the shell behaves the same as a point charge for points in space outside the sphere, and doesn't effect a field inside.