1.54 $\vec{V} = r^2\cos\theta\,\hat{r} + r^2\cos\phi\,\hat{\theta} - r^2\cos\theta\sin\phi\,\hat{\phi}$

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

$$\frac{\partial}{\partial r}(r^2 v_r) = \frac{\partial}{\partial r}(r^4\cos\theta) = 4r^3\cos\theta \Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) = 4r\cos\theta$$

$$\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) = \frac{\partial}{\partial\theta}(r^2\sin\theta\cos\phi) = r^2\cos\theta\cos\phi \Rightarrow \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) = r\cot\theta\cos\phi$$

$$\frac{\partial v_\phi}{\partial\phi} = \frac{\partial}{\partial\phi}(-r^2\cos\theta\sin\phi) = r^2\cos\theta\cos\phi \Rightarrow -\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} = -r\cot\theta\cos\phi$$

$$\vec{\nabla}\cdot\vec{V} = 4r\cos\theta + r\cot\theta\cos\phi - r\cot\theta\cos\phi = 4r\cos\theta$$

$$\int_0^R\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} 4r\cos\theta\, r^2\sin\theta\, d\theta\, d\phi\, dr = \frac{R^4}{2}\left(\frac{\pi}{2}\right)\underbrace{\int_0^{\frac{\pi}{2}}\sin(2\theta)\,d\theta}_{=1} = \frac{\pi R^4}{4}$$

4 surfaces. Start with sphere

then $d\vec{a} = r^2 d\Omega\,\hat{r} = r^2\sin\theta\, d\theta\, d\phi$ so $\vec{V}\cdot d\vec{a} = r^4\cos\theta\sin\theta$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} R^4\cos\theta\sin\theta\, d\theta\, d\phi = \frac{\pi}{2}R^4\left[\frac{-1}{4}\cos 2\theta\right]_0^{\frac{\pi}{2}} = \frac{\pi R^4}{4}$$

xy plane, use polar coordinates, for which $\theta = \frac{\pi}{2}$

$d\vec{a} = -r\,dr\,d\phi\,\hat{k}$

$\vec{V}\cdot d\vec{a} = -[r^3\cos^2\theta - r^3\cos\phi\sin\theta]\,dr\,d\phi$ (see eq. 64)

$$\int_0^{\frac{\pi}{2}}\int_0^R r^3\cos\phi\, dr\, d\phi = \frac{1}{4}R^4[\sin\phi]_0^{\pi/2} = \frac{R^4}{4}$$

xz plane: $d\vec{a} = -r\,d\phi\,dr\,\hat{j}$, $\phi = 0$

$$\vec{V}\cdot d\vec{a} = -[r^3\cos\theta\sin\theta\sin\phi + r^3\cos\phi\sin\phi\cos\theta - r^3\cos\theta\sin\phi\cos\phi] = 0$$

1.54 yz-plane: $\phi = \frac{\pi}{2}$

$d\vec{a} = -r\,d\theta\,dr\,\hat{x}$

$\vec{v}\cdot d\vec{a} = -r\,d\theta\,dr\left[r^2\cos\theta\sin\theta\cos\phi + r^2\cos^2\phi\cos\theta + r^2\cos\theta\sin^2\phi\right] = -r^3\cos\theta$

$$\int_S \vec{v}\cdot d\vec{a} = -\int_0^R\int_0^{\frac{\pi}{2}} r^3\cos\phi\,d\phi\,dr = \frac{-R^4}{4}$$

$$\Rightarrow \oint_S \vec{v}\cdot d\vec{a} = \frac{\pi R^4}{4} + 0 + \frac{R^4}{4} - \frac{R^4}{4} = \frac{\pi R^4}{4}$$