Damped Oscillators in the Time Domain

April 26, 2023

Damped oscillators are any system whose time evolution can be described by an equation of the form

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t) \tag{1}$$

where x is the system's quantity of interest, t is time, F is an arbitrary forcing function, and m, b, and k are constants of the system.

There are two special cases I would like to look at in this post. The first is when there is no forcing function, $F(t) = 0$. The second is when the forcing function takes the form $F(t) = F_0 \cos(\omega t)$. These two cases make up the vast majority of use cases in practice.

1 Free Oscillation

If $F(t) = 0$ we then have what is known as free oscillations. This results in the homogeneous equation

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0. \tag{2}$$

We can begin with the classic trial solution $x = e^{rt}$. From here we have

$$\begin{aligned} x &= e^{rt}, \\ \frac{dx}{dt} &= re^{rt}, \text{ and} \\ \frac{d^2x}{dt^2} &= r^2e^{rt} \end{aligned}$$

Inserting these into eq. 2 yields

$$mr^2e^{rt} + bre^{rt} + ke^{rt} = 0$$

and from here we can divide through by e^{rt} to arrive at the characteristic equation (also sometimes called the auxiliary equation):

$$mr^2 + br + k = 0 \tag{3}$$

which has two solutions:

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}. \tag{4}$$

Of critical interest is the discriminant $b^2 - 4mk$. This quantity is important enough to be given its own variable, $\Delta = b^2 - 4mk$. The value of Δ determines the large-scale behavior of the system, which can be broken into three regimes: $\Delta > 0$, $\Delta = 0$, and $\Delta < 0$.

1.1 Overdamped Regime

$\Delta > 0$ is called the the overdamped regime. For a positive discriminant, the roots of the characteristic equation are real valued and distinct. This gives us two linearly independent solution functions to the ODE. Any linear combination of these two functions will thus be a solution to the ODE. The most general solution to the homogeneous equation is then simply[1]

$$x(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t} \tag{5}$$

where A_1 and A_2 are constants determined by the initial conditions. Let's pick a specific set of initial conditions and see how the system behaves in each regime. For simplicity's sake let's pick $x(0) = 2$, $\dot{x}(0) = 0$. We can now determine A_1 and A_2 as follows:

$$\begin{aligned} x(0) &= A_1 + A_2 = 2 \rightarrow A_2 = 2 - A_1 \\ \dot{x}(0) &= A_1 r_1 + A_2 r_2 = 0 \end{aligned}$$

We can substitute the first equation into the second to get

$$A_1 r_1 + A_2 r_2 = A_1 r_1 + (2 - A_1) r_2 = A_1(r_1 - r_2) + 2r_2 = 0 \rightarrow A_1 = -\frac{2r_2}{r_1 - r_2}$$

And now substituting this back into the first equation,

$$A_2 = 2 - \frac{-2r_2}{r_1 - r_2} = \frac{2r_1 - 2r_2 + 2r_2}{r_1 - r_2} = \frac{2r_1}{r_1 - r_2}.$$

So,

$$x(t) = \left(\frac{2}{r_1 - r_2}\right)(r_1 e^{r_2 t} - r_2 e^{r_1 t}). \tag{6}$$

Next we need to pick values of m, b, and k so that Δ is positive. Let's choose $m = 2$, $b = 40$, and $k = 50$. Then $\Delta = 1200 > 0$. We can now plot $x(t)$ for several seconds to see its behavior:

[1] This is because we are solving a *second* order linear ODE. A linear ODE's general solution requires the same number of linearly independent functions as its order. The reason for this is beyond the scope of the current discussion. Any introductory ODE text will cover this.

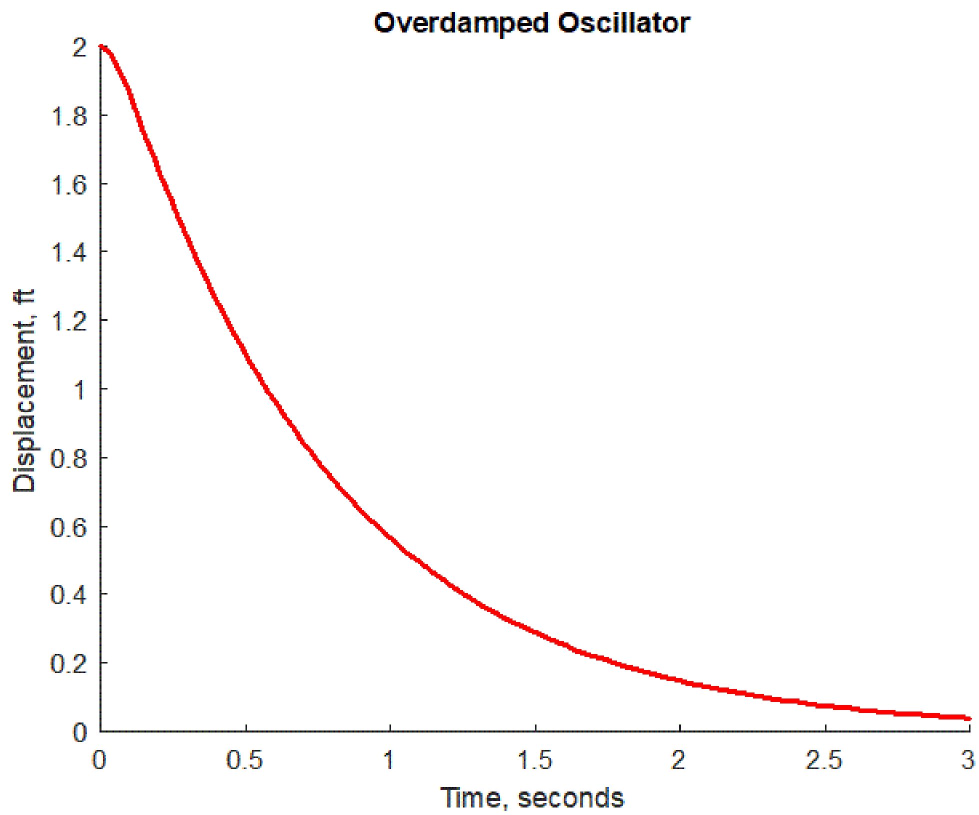

Figure 1: Example of an overdamped system. The system decays back to equilibrium without oscillations. The damping coefficient is so strong it prevent the system from ever overshooting equilibrium.

1.2 Critically Damped Regime

The critically damped regime occurs when $\Delta = 0$. If this is the case, then $r_1 = r_2 = -b/2a$. Let $r = -b/2a$. Now we cannot use the same solution as for the overdamped regime because we would have

$$x(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t} = A_1 e^{rt} + A_2 e^{rt} = (A_1 + A_2) e^{rt}$$

which shows that here $x(t)$ is a single function. To ensure our solution covers all possible cases, we need $x(t)$ to be a linear combination of two linearly independent functions. We must hunt around for another trial solution. The next simplest function to try is te^{st}. Then we have

$$\begin{aligned} x(t) &= te^{st} \\ \dot{x}(t) &= e^{st} + ste^{st} \text{ and} \\ \ddot{x}(t) &= se^{st} + se^{st} + s^2te^{st} = 2se^{st} + s^2te^{st}. \end{aligned}$$

We can now substitute these equations back in to equation (2) to get

$$\begin{aligned} m(2se^{st} + s^2te^{st}) + b(e^{st} + ste^{st}) + kte^{st} &= 0 \\ \rightarrow m(2s + s^2t) + b(1 + st) + kt &= 0 \\ \rightarrow (2ms + b) + (ms^2 + bs + k)t &= 0 \end{aligned}$$

The trial function must be valid for all $t \geq 0$. Since t is essentially always non-zero (except at one point), this means both the terms in parenthesis must separately vanish. There is no guarantee however that this will or even *can* occur. We thus have the following two requirements:

$$\begin{aligned} 2ms + b &= 0 \\ ms^2 + bs + k &= 0 \end{aligned}$$

The second equation we recognize as the characteristic equation which requires $s = (-b \pm \sqrt{b^2 - 4mk})/2m)$. And for critically damping the discriminant is zero so this becomes

$$s = -b/2m.$$

From the first equation we have

$$s = -b/2m$$

so indeed, our trial function is a solution with $s = r$. Notice this function is a solution *only* for the critically damped case. For any other case, the discriminant is non-zero and so we would have a contradiction on the value of s.

Putting this all together we have as general solution for critical damping

$$x(t) = B_1e^{rt} + B_2te^{rt} = (B_1 + B_2t)e^{rt} \tag{7}$$

Let's determine the solution for the particular initial conditions $x(0) = 2$ and $\dot{x}(0) = 0$. Then using the fact that $\dot{x}(t) = B_2e^{rt} + r(B_1 + B_2t)e^{rt}$ we have

$$\begin{aligned} x(0) = B_1 &\rightarrow B_1 = 2 \\ \dot{x}(0) = B_2 + rB_1 &\rightarrow B_2 = -rB_1 = -2r \\ &\rightarrow x(t) = 2r(1 - t)e^{rt} \end{aligned}$$

We can now take the example system from the section on overdamped oscillators and dial the damping coefficient down to $b = 20$ so that combined with $m = 2$ and $k = 50$ we have $\Delta = 400 - 400 = 0$ as desired. Plotting this out for several seconds we get the following curve:

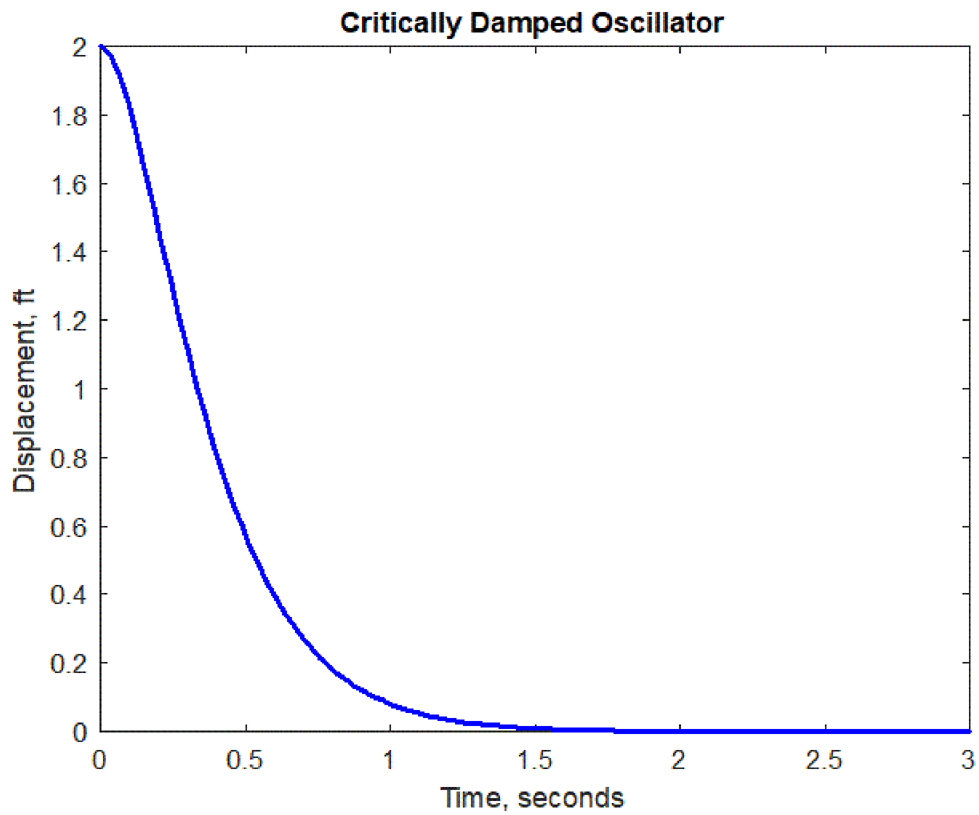

Figure 2: Example of a critically damped system response. Compared to figure **??**, this curve decays much faster back to equilibrium.

1.3 Underdamped Regime

If $\Delta < 0$ we are in the underdamped regime. Now the roots of the characteristic equation are not real but a conjugate pair of complex numbers viz.

$$r_{1,2} = \frac{-b}{2m} \pm i\frac{\sqrt{4mk - b^2}}{2m}$$

We again have two linearly independent solutions to eq. (2) so we can immediately arrive at the general solution:

$$x(t) = e^{\frac{-b}{2m}t}\left(C_1 e^{i\frac{\sqrt{-\Delta}}{2m}t} + C_2 e^{-i\frac{\sqrt{-\Delta}}{2m}t}\right)$$

There is still quite a bit of clean-up to do before we can reach the more commonly used form of the solution. First let's define some new variables to tidy up notation. Let $\tau = 2m/b$ and $\omega = \sqrt{-\Delta}/2m$. Now we may write x as

$$x(t) = e^{-t/\tau}(C_1 e^{i\omega t} + C_2 e^{-i\omega t})$$

Keep in mind here that $e^{i\omega t}$ can in general be a complex number. But $x(t)$ must always be a real number! This means C_1 and C_2 must also be complex in just the right way to ensure the imaginary components of $e^{\pm i\omega t}$ always cancel out for all choices of t. This seems like a rather daunting problem to work out, but introducing a few new variables let's us sweep everything under the rug.

First, let's notice that since $x(t)$ is a real number, it is equal to its complex conjugate. Therefore we have

$$x(t) = x(t)^* \rightarrow C_1 e^{i\omega t} + C_2 e^{-i\omega t} = C_1^* e^{-i\omega t} + C_2^* e^{i\omega t}$$

Now $e^{i\omega t}$ and $e^{-i\omega t}$ are linearly independently functions, so the only way this relation is true is if the coefficients in front of them are the same for both sides of the equation. This implies that

$$\begin{aligned} C_1 &= C_2^* \text{ and} \\ C_2 &= C_1^* \end{aligned}$$

The second relation is simply the complex conjugate of the first and so provides no new information. Defining $f(t) = e^{i\omega t}$, We can then write

$$C_1 e^{i\omega t} + C_2 e^{-i\omega t} = C_1 f(t) + C_1^* f(t)^* = C_1 f(t) + (C_1 f(t))^*$$

And for all choices of t this boils down to some complex number plus its conjugate, which is always real[2]. It might seem that perhaps we've made an error because a second order ODE requires two constants of integration and now we only have 1, C_1. But despair not! For C_1 is complex, so it may be written as $C_1 = a + ib$ which contains two distinct constants.

Returning to $x(t)$, now we can make use of Euler's formula and define some new constants $D_1 = C_1 + C_2$ and $D_2 = i(C_1 - C_2)$. (Notice both these constants will be real numbers since $z + z^* = 2\mathfrak{Re}\{z\}$ and $z - z^* = 2i\mathfrak{Im}\{z\}$.) So now we have

$$x(t) = e^{-t/\tau}\left(D_1 \cos(\omega t) + D_2 \sin(\omega t) \right).$$

This is looking much cleaner, but there is one final redefinition of constants we can make to clean this expression up even further.

[2]This can easily be seen by considering an arbitrary complex number $z = a + ib$. Then $z^* = a - ib$ and so $z + z^* = 2a$ which is real.

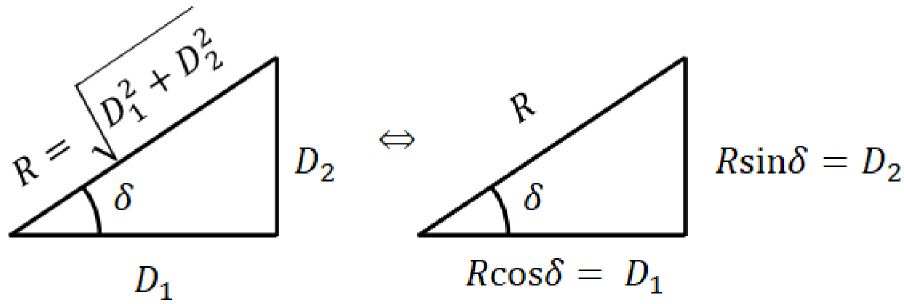

Figure 3: D_1 and D_2 can be used to form the legs of a triangle. This lets us turn our integrating constants into a Radius-angle pair.

Consulting figure 3, We can let D_1 and D_2 form the legs of a right triangle. Then we can define the new variables R and δ as $R = \sqrt{D_1^2 + D_2^2}$ and $\delta = \arctan(D2/D1)$. More useful to us are the inverse relations $D_1 = R\cos\delta$ and $D_2 = R\sin\delta$ which we can substitute into $x(t)$ to arrive at

$$x(t) = e^{-t/\tau}(R\cos\delta\cos\omega t + R\sin\delta\sin\omega t).$$

Finally we can make use of the trig. identity $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ to arrive at the desired result:

$$x(t) = e^{-t/\tau}(\,R\cos(\omega t - \delta)\,). \tag{8}$$

Let's see how how our test system behaves if we use the same initial conditions and dial our damping down to $b = 7$ (again with $m = 2$ and $k = 50$). Now we have

$$\dot{x}(t) = -R\omega\sin(\omega t - \delta)$$

so,

$$\begin{aligned}
&x(0) = 2 = R\cos\delta \\
&\dot{x}(0) = 0 = -\omega R\sin(-\delta) \to \sin(\delta) = 0 \to \delta = 0 \\
&\to R = 2
\end{aligned}$$

So our test system's motion is governed by

$$x(t) = 2\cos(\omega t).$$

Plotting this out we get the following:

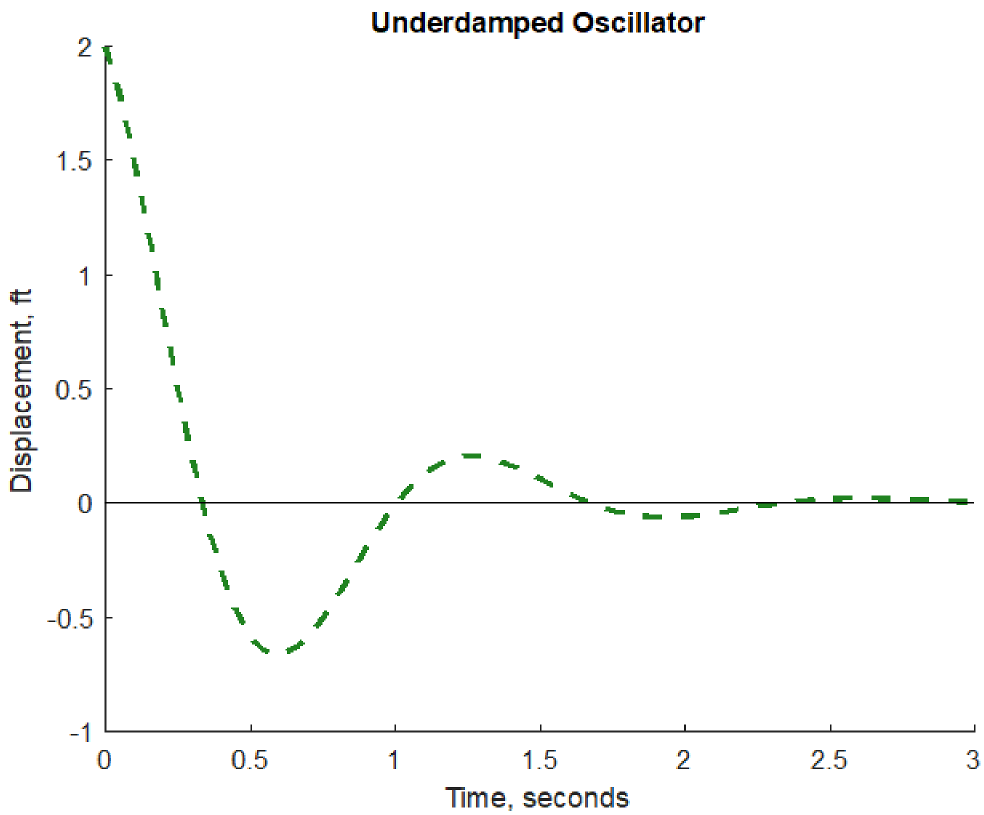

Figure 4: In the underdamped regime, the damping coefficient is no longer strong enough to stop the system from over shooting. Now the system oscillates about its equilibrium before settling down.

1.4 Special case: no damping

If $b = 0$ there is no damped motion, and so the characteristic equation becomes $mr^2 + k = 0$. In this case r_1 and r_2 are not just complex but purely imaginary. Namely,

$$r_{1,2} = \pm i\sqrt{\frac{k}{m}}$$

so we define $\omega_0 = \sqrt{k/m}$. From here the solution is identitical to the case for underdamped systems. The only difference is that now there is no $e^{-t/\tau}$ in front of the final solution so the oscillations will not decay. Instead, we have for our general solution

$$x(t) = R\cos(\omega t - \delta). \tag{9}$$

ω_0 is often called the natural frequency because it is the frequency a system will oscillate at if not damped and left to its own devices (i.e. not driven). This notation is used even in damped systems as we will see in the next subsection.

1.5 The damping ratio

The turning point from oscillatory to non-oscillatory behavior in the system occurs at $\Delta = 0$. Put another way, the turning point is when $b^2 = 4mk$. Thus it is the balance between b^2 and $4mk$ that determines the large scale behavior of the system. Since the sign of Δ depends on the relative size of the two quantities b^2 and $4mk$, we are really interested in their ratio. Instead of looking at the discriminant we could define a new unitless quantity $b^2/4mk$. Critical damping occurs when this ratio is 1, while overdamped behavior occurs when the ratio is greater than 1 because this implies b^2 is larger than $4mk$. Likewise, when the ratio is below 1 the system is underdamped. In practice however, it is not this ratio, but rather its square root $\zeta = b/2\sqrt{mk}$, that is used. This is called the damping ratio.

The great utility of ζ is that we can rewrite eq.'s (1) and (2) entirely in terms of ζ and ω_0 (and of course x and its derivatives). We begin by substituting $b = 2\zeta\sqrt{mk}$ then dividing through by k.

$$\frac{m}{k}\frac{d^2x}{dt^2} + \frac{2\zeta\sqrt{mk}}{k}\frac{dx}{dt} + x = \frac{1}{k}F(t)$$

Next we simplify and substitute in ω_0 where possible, noting that $1/\omega_0^2 = k/m$

$$\frac{1}{\omega_0^2}\frac{d^2x}{dt^2} + \frac{2\zeta\sqrt{mk}}{m}\frac{dx}{dt} + kx = \frac{1}{k}F(t)$$
$$\frac{1}{\omega_0^2}\frac{d^2x}{dt^2} + 2\zeta\sqrt{\frac{m}{k}}\frac{dx}{dt} + x = \frac{1}{k}F(t)$$

Recognizing that since k is constant we can absorb it into the forcing function, we arrive at the alternate form of expressing the system as

$$\frac{1}{\omega_0^2}\frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_0}\frac{dx}{dt} + x = \mathfrak{F}(t) \tag{10}$$

where $\mathfrak{F}(t) = F(t)/k$. For undriven, underdamped systems, we can express the frequency the system will oscillate at, $\omega = \frac{\sqrt{4mk-b^2}}{2m}$, in terms of ζ and ω_0:

$$\begin{aligned}
\omega &= \frac{\sqrt{4mk-b^2}}{2m} \\
&= \frac{(2\sqrt{mk})\sqrt{1-b^2/4mk}}{2m} \\
&= \sqrt{\frac{k}{m}}\sqrt{1-\zeta^2} \\
\omega &= \omega_0\sqrt{1-\zeta^2}
\end{aligned}$$

Let's see how the system behaves for each distinct region of ζ:

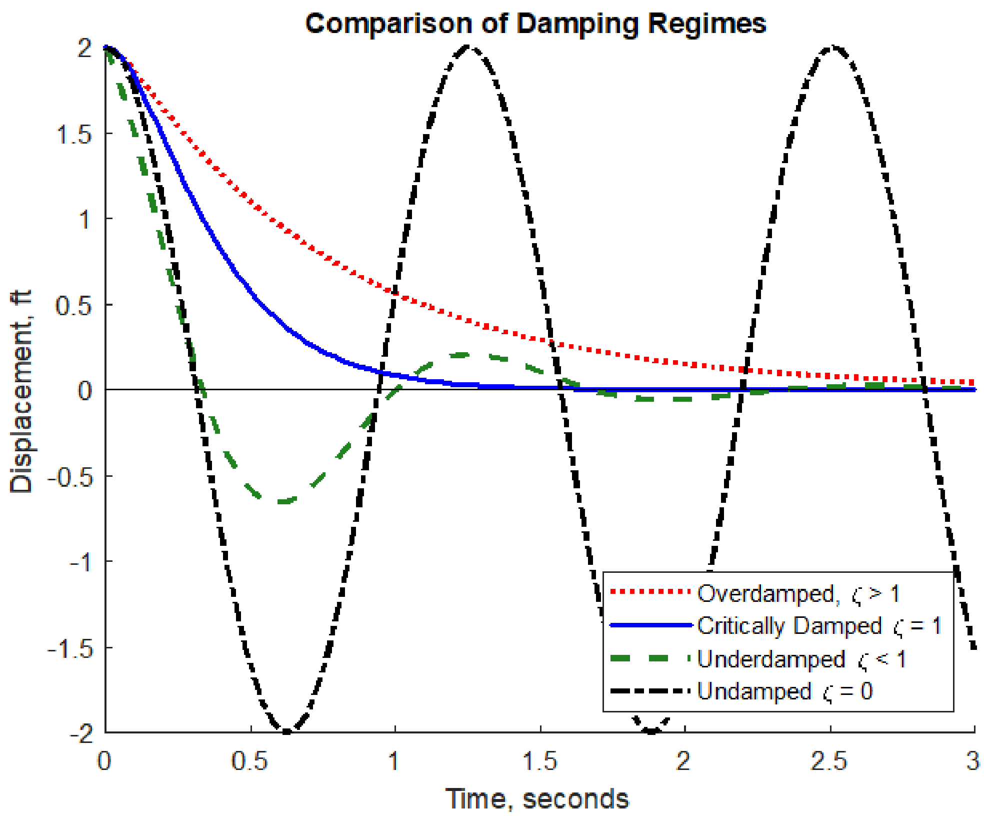

Figure 5: The four distinct behaviors of the system for varying ζ

2 Driven Damped Oscillator

Now that we fully understand the homogeneous equation to eq. (1), it's time to solving eq. (1) for the case of $F(t) = F_0 \cos(\omega_d t)$. Explicity, we aim to solve the equation

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 cos(\omega_d t). \tag{11}$$

As it turns out though, thanks to Euler's formula, it's easier to solve this other, similar equation instead:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 e^{i\omega_d t} \tag{12}$$

If we take just the real part of both sides of this equation, we recover eq. (11) so a solution to eq. (12) is just as useful to us.

We solve this equation in nearly the same manner as before. We start with the trial solution $x(t) = Ce^{i\omega_d t}$ with C a complex constant. In fact, it's helpful to rewrite this constant in polar form as $Re^{i\phi}$ then we have

$$
\begin{aligned}
x(t) &= Re^{i(\omega_d t+\phi)} \\
\dot{x}(t) &= iR\omega_d e^{i(\omega_d t+\phi)} \\
\ddot{x}(t) &= -R\omega_d^2 e^{i(\omega_d t+\phi)}.
\end{aligned}
$$

Substitution back into eq. (12) gives us

$$-m\omega_d^2 Re^{i(\omega_d t+\phi)} + ibR\omega_d e^{i(\omega_d t+\phi)} + kRe^{i(\omega_d t+\phi)} = F_0 e^{i\omega_d t}$$

and dividing through by $e^{i(\omega_d t+\phi)}$ yields

$$
\begin{aligned}
-m\omega_d^2 R + ibR\omega_d + kR &= F_0 e^{-i\phi} \\
R(-m\omega_d^2 + ib\omega_d + k) &= F_0 e^{-i\phi} \\
R &= \frac{F_0 e^{-i\phi}}{k - m\omega_d^2 + ib\omega_d}
\end{aligned}
$$

But we know R *must* be a real number for this to be a valid solution. Is this possible or have we arrived at a contradiction? The fact that we are free to choose ϕ as necessary saves us from a contradiction. We need only to rewrite the denominator in polar form and then we can choose ϕ to be equal to the denominator's argument. A picture will help to explain things here.

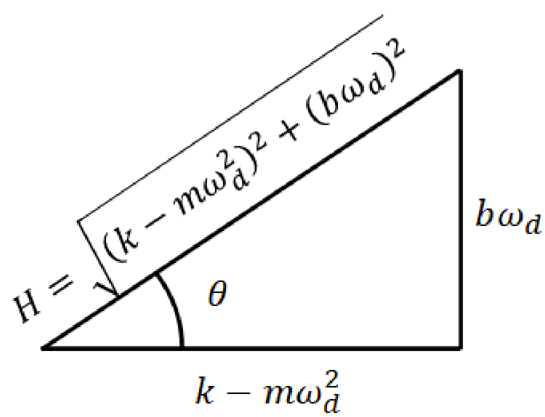

Figure 6: Visualizing the denominator in the complex plane.

From figure 6 we can see that the denominator can be converted to polar form as $He^{i\theta}$ where

$$H = \sqrt{(k - m\omega_d^2)^2 + (b\omega_d)^2} = \sqrt{m^2(\omega_0^2 - \omega_d^2)^2 + (b\omega_d)^2}$$
$$\theta = \arctan\left(\frac{b\omega_d}{k - m\omega_d^2)}\right) = \arctan\left(\frac{b\omega_d}{m(\omega_0^2 - \omega_d^2)}\right)$$

and we have used the fact that $k/m = \omega_0^2$ to factor out m where appropriate. Then we have

$$R = \frac{F_0 e^{-i\phi}}{He^{i\theta}} = \frac{F_0}{H} e^{i(\theta - \phi)}$$

and since R must be real, then it must be that $\phi = \theta$. This gives us

$$\phi = \theta = \arctan\left(\frac{b\omega_d}{m(\omega_0^2 - \omega_d^2)}\right)$$
$$R = \frac{F_0}{H} = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega_d^2)^2 + (b\omega_d)^2}}$$
$$\rightarrow x(t) = Re^{i(\omega_d t + \phi)} = \frac{F_0 e^{i(\omega_d t + \theta)}}{\sqrt{(k - m\omega_d^2)^2 + (b\omega_d)^2}}.$$

Then to get the solution to our original problem, eq. (11), we simply take the real part of this solution to arrive at

$$x(t) = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega_d^2)^2 + (b\omega_d)^2}} \cos(\omega_d t + \theta) \tag{13}$$

where we have used the fact that $\omega_0^2 = k/m$ to factor out the m from the first term in the denominator.

What's worth noting in this solution is that the amplitude will change in size depending on the drive frequency. The phase also depends on the drive frequency. In general, the way the frequency and phase of a system depends on the driving force's frequency is called the "frequency response" of the system. Plotting both the amplitude and phase as a function of the driving frequency allows for a nice way to visualize the frequency response and is called a Bode plot. The bode plot for the underdamped system studied in section 1.3 is shown in figure 7.

2.1 Resonance

Let's return to the amplitude of $x(t)$ in eq. (13). The maximum will occur when the denominator is a minimum. We can find this minimum by taking the derivative of the denominator with respect to ω_d and setting it equal to zero.

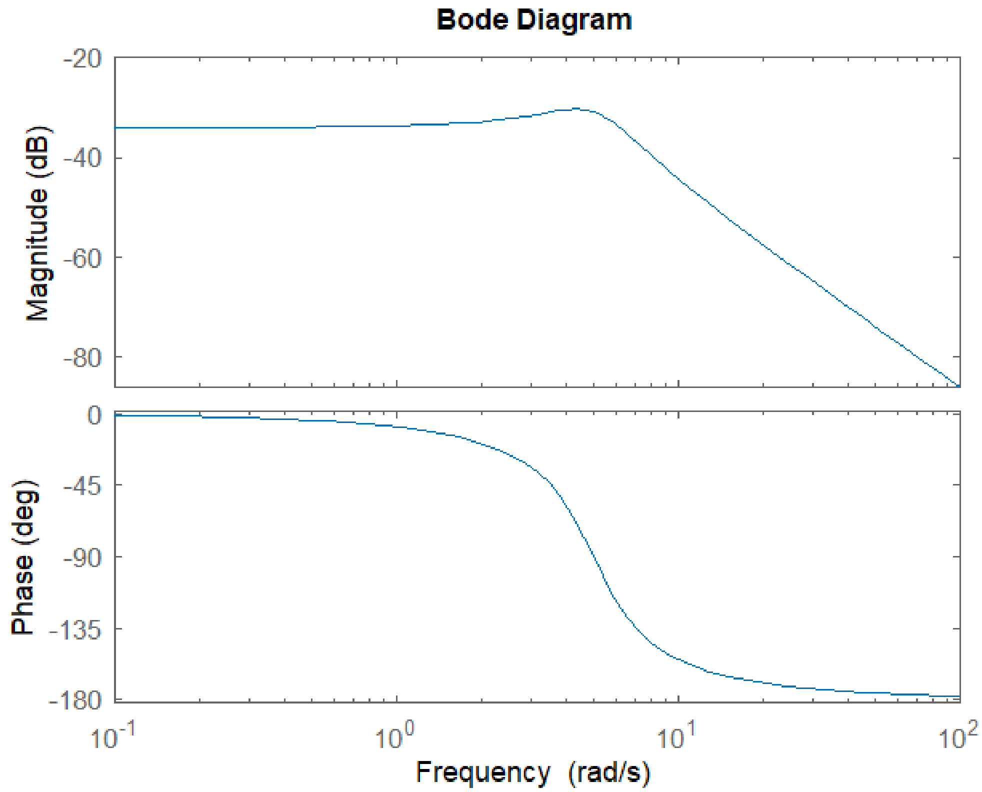

Figure 7: Bode plot for the system with $m = 2$, $b = 7$, $k = 50$.

In practice though, it's easier to take the derivative of what's inside the square root only[3]:

$$\frac{d}{d\omega_d}\left[(m^2(\omega_0^2 - \omega_d)^2 + b^2\omega_d^2)\right] = (-2\omega_d)m^2 \cdot 2(\omega_0^2 - \omega_d^2) + 2b^2\omega_d = 0$$
$$= 2\omega_d(b^2 - 2m^2(\omega_0^2 - \omega_d^2)).$$

There are two solutions, one of which is $\omega_d = 0$ which is nonphysical and can be dispensed with leaving us with the resonance frequency:

$$\omega_{d,r} = \sqrt{\omega_0^2 - \frac{b^2}{2m^2}} = \omega_0\sqrt{1 - 2\zeta^2} \tag{14}$$

[3]There are several ways to see why this is a valid thing to do. One is to consider the chain rule. Beginning with $\sqrt{u}$, the derivative will give us $(u)^{-1/2}u'$. Since we are setting this equal to zero we can multiply through by $u^{1/2}$ leaving us with just u' which is the derivative of what's inside the square root. Another way is to recognize that the square root is a monotonically increasing function, so the minimum of what's inside the square root will be the square root's minimum as well.

Returning to figure 7, we can see this resonance frequency manifests itself as the bump in the amplitude curve around 4.5 Hz. Notice the resonance frequency is very similar in form to the ringning frequency of an undriven system ($\omega_0\sqrt{1-\zeta^2}$). What's curious to notice though, is that while there always exists a ringing frequency for any underdamped system when undriven, there are certain underdamped systems that when driven have no resonance frequency. For it is apparent from eq. (14) that if $\zeta > \sqrt{1/2}$ then there will be no resonance in the system. For instance, choosing a system with $\zeta = 0.8$, we can see from figure 8 that there is no bump in the bode amplitude plot at any frequency.

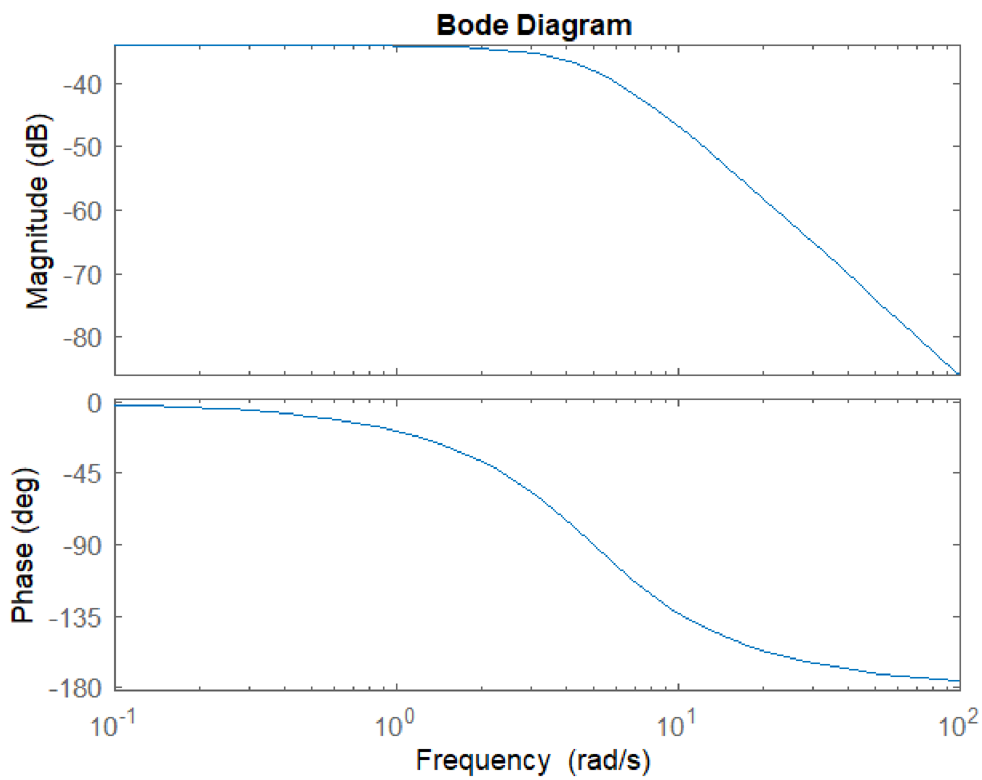

Figure 8: Bode plot for the system with $\zeta = 0.8$, $m = 2$, $k = 50$.

Let's see how a system's resonance behaves as we vary the damping coefficient:

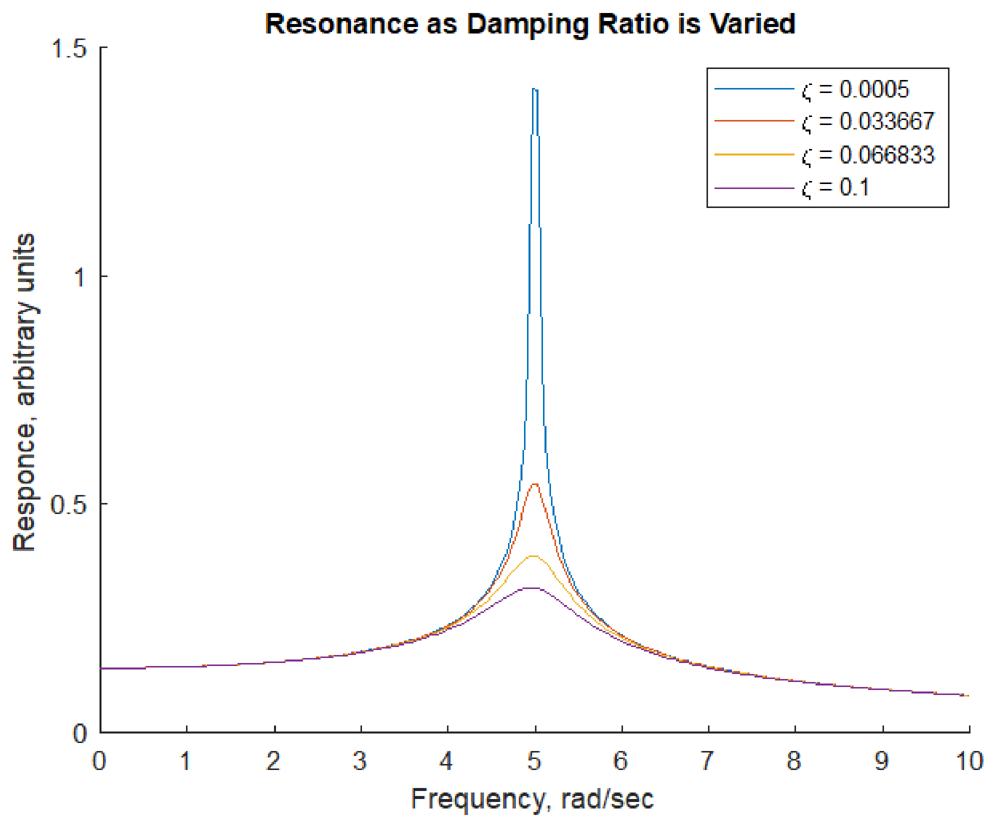

Figure 9: Resonance as a function of frequency for $m = 2$, $k = 50$, ζ varied.

We can see that resonance becomes more extreme as the damping of the system is lowered. We can see also that the peaks become thinner as the resonance's strength increases. Just like we have the damping ratio to characterize whether a system will exhibit oscillatory behavior or not, it would be nice to have a quantity that gives us some information about the quality of our resonator, i.e. how sharply peaked it is. The previous plot may make it seem that the damping ratio tells us this information, but that is not so. Figure 10 below shows the comparison of 3 different systems all with the same damping ratio of $\zeta = .0625$. We can plainly see there is still something different between these resonators despite their having the same damping ratio. Specifically, we can see a trend that as the resonance frequency decreases, the peak sharpens. This leads us to what is called the resonator's quality factor or Q-factor, which is a ratio of the amplitude response's resonance frequency to it's width. And by width, I specifically mean the "full width at half maximum" (FWHM in the literature). The full width at half maximum is the length of the frequency range over which the amplitude response is at least half the maximum value or more (see fig 11).

$$Q = \frac{\omega_{d,r}}{FWHM} \tag{15}$$

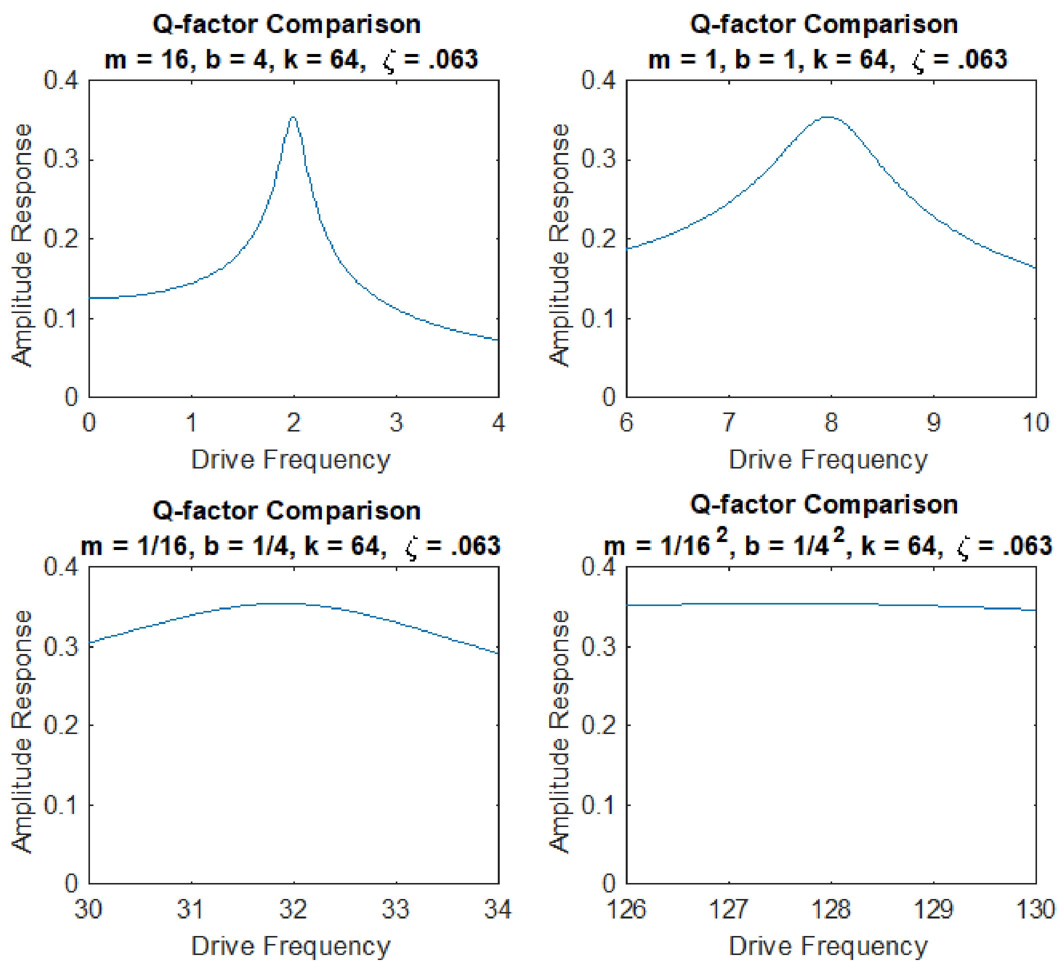

Figure 10: Comparison of resonance for various system all with $\zeta = 0.0625$.

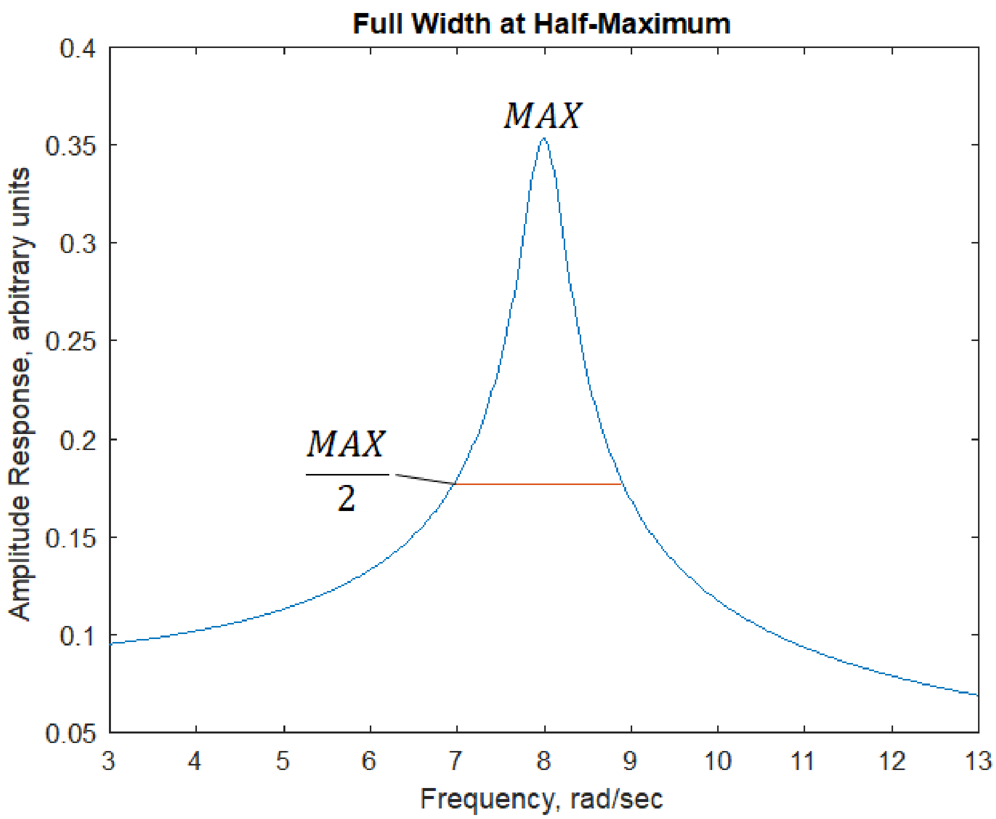

Figure 11: Visualization of the FWHM (orange line) of a resonator.