1.23 The dimensionality of the vector space is 3. Since B could only have eigenvalues b or −b, it is also degenerate. However combined w/ the eigenvalues a & −a from A, these could form a complete basis.

$$AB = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix}, \quad BA = \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix}$$

So $[A, B] = 0$

We require eigenkets $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ s.t.

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda_a \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda_b \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

so

$$\begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{matrix} bx = \lambda_b x \Rightarrow \lambda_b = \pm b \\ \left.\begin{matrix} -ibz = \lambda_b y \\ iby = \lambda_b z \end{matrix}\right\} \Rightarrow y = \pm iz \end{matrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ \pm iz \\ z \end{bmatrix}$$

the easy choices are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$

as a check

$$A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \frac{A}{\sqrt{2}}\begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} = \frac{-a}{\sqrt{2}}\begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}, \quad A\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} = \frac{-a}{\sqrt{2}}\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

$$B\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = b\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \frac{B}{\sqrt{2}}\begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -ib \\ -i^2 b \end{bmatrix} = \frac{b}{\sqrt{2}}\begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}, \quad \frac{B}{\sqrt{2}}\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -ib \\ i^2 b \end{bmatrix} = \frac{-b}{\sqrt{2}}\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

So the eigenkets are $|a, b\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; |-a, b\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$

$|-a, -b\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ i \\ \end{bmatrix}$