1.11

$$H \doteq \begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{bmatrix} \Rightarrow \det \begin{bmatrix} H_{11}-h & H_{12} \\ H_{12} & H_{22}-h \end{bmatrix} =$$

$$= (H_{11}-h)(H_{22}-h) - H_{12}^2 = H_{11}H_{22} - (H_{11}+H_{22})h + h^2 - H_{12}^2 = 0$$

$$\Rightarrow h^2 - (H_{11}+H_{22})h + H_{11}H_{22} - H_{12}^2 = 0$$

$$\Rightarrow h = \frac{H_{11}+H_{22} \pm \sqrt{(H_{11}+H_{22})^2 - 4(H_{11}H_{22}-H_{12}^2)}}{2}$$

$$(H_{11}+H_{22})^2 - 4(H_{11}H_{22}-H_{12}^2) = H_{11}^2 + 2H_{11}H_{22} + H_{22}^2 - 4H_{11}H_{22} + 4H_{12}^2$$

$$= (H_{11}-H_{22})^2 + 4H_{12}^2$$

$$\Rightarrow h_{1,2} = \frac{H_{11}+H_{22} \pm \sqrt{(H_{11}-H_{22})^2 + 4H_{12}^2}}{2}$$

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = h_{1,2} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow H_{11}x + H_{12}y = h_{1,2}x$$

$$\Rightarrow x = \frac{H_{12}}{h_{1,2}-H_{11}} y \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} \frac{H_{12}}{h_{1,2}-H_{11}} \\ 1 \end{bmatrix}$$

$$y^2\left(1^2 + \left(\frac{H_{12}}{h_{1,2}-H_{11}}\right)^2\right) = 1 = y^2\left(1^2 + \frac{H_{12}^2}{h^2 - 2hH_{11} + H_{11}^2}\right)$$

$$= \left(\frac{h^2 - 2hH_{11} + H_{11}^2 + H_{12}^2}{(h^2-H_{11})^2}\right) \Rightarrow y = \frac{h - H_{11}}{\sqrt{h^2 - 2hH_{11} + H_{11}^2 + H_{12}^2}}$$

so

$$|h_{1,2}\rangle = |2\rangle\left(\frac{h_{1,2}-H_{11}}{\sqrt{h_{1,2}^2 - 2hH_{11} + H_{11}^2 + H_{12}^2}}\right) + \left(\frac{H_{12}}{\sqrt{h_{1,2}^2 - 2h_{1,2}H_{11} + H_{11}^2 + H_{12}^2}}\right)|1\rangle$$

however, if $H_{1,2} = 0$, then $|h_1\rangle$ and $|h_2\rangle$ are not linearly independt. this suggest that for $|h_2\rangle$ we try

$$H_{12}x + H_{22}y = h_2 y \Rightarrow y = \frac{H_{12}x}{h_2 - H_{22}}$$

$$\Rightarrow x\begin{bmatrix} 1 \\ \dfrac{H_{12}}{h_2 - H_{22}} \end{bmatrix} \Rightarrow x^2\left(1^2 + \left(\frac{H_{12}}{h_2 - H_{22}}\right)^2\right) = 1$$

$$= x^2\left(\frac{h_2^2 - 2h_2 H_{22} + H_{22}^2 + H_{12}^2}{(h_2 - H_{22})^2}\right) \Rightarrow x = \frac{(h_2 - H_{22})}{\sqrt{(h - H_{22})^2 + H_{12}^2}}$$

so

$$|h_2\rangle = \frac{(h_2 - H_{22})}{\sqrt{(h - H_{22})^2 + H_{12}^2}}|1\rangle + \frac{H_{12}}{\sqrt{(h_2 - H_{22})^2 + H_{12}}}|2\rangle$$

to reiterate

$$|h_1\rangle = \frac{H_{12}}{\sqrt{(h_1 - H_{11})^2 + H_{12}^2}}|1\rangle + \frac{(h_1 - H_{11})}{\sqrt{(h_1 - H_{11})^2 + H_{12}}}|2\rangle$$

If $H_{12} = 0$ then

$|h_1\rangle = |2\rangle$ and $|h_2\rangle = |1\rangle$

as expected since H in this case is already digonal in the $|1\rangle, |2\rangle$ basis.