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1.10 Since H is characterized by 2 kets, we expect H to have 2 eigenkets, $|h_1\rangle$ and $|h_2\rangle$, satisfying the relation

$$H|h_1\rangle = h_1|h_1\rangle.$$

So

$$\sum_i \langle j|H|i\rangle\langle i|h_1\rangle = h_1 \langle j|h_1\rangle.$$

Thus, $h_1\langle 1|h_1\rangle = \langle 1|H|1\rangle\langle 1|h_1\rangle + \langle 1|H|2\rangle\langle 2|h_1\rangle$

$\langle 1|H|1\rangle = a$, $\langle 1|H|2\rangle = a$,

so

$$(h_1-a_1)\langle 1|h_1\rangle = a\langle 2|h_1\rangle \Rightarrow \langle 2|h_1\rangle = \frac{h_1-a}{a}\langle 1|h_1\rangle$$

$$|h_1\rangle = |1\rangle\langle 1|h_1\rangle + |2\rangle\langle 2|h_1\rangle = |1\rangle\langle 1|h_1\rangle + \left(\frac{h_1-a}{a}\right)\langle 1|h_1\rangle|2\rangle$$

$$\langle h_1|h_1\rangle = 1 = |\langle 1|h_1\rangle|^2 + \langle h_1|2\rangle\langle 1|h_1\rangle\left(\frac{h_1-a}{a}\right)$$

$$1 = |\langle 1|h_1\rangle|^2 + \left(\frac{h_1-a}{a}\right)^2|\langle 1|h_1\rangle|^2 = |\langle 1|h_1\rangle|^2\left(\frac{a^2+h_1^2-2ah_1+a^2}{a^2}\right)$$

Choosing the phase to be real,

$$\langle 1|h_1\rangle = \frac{a}{\sqrt{h_1^2-2ah_1+2a^2}} = \frac{a}{\lambda_1} \quad \text{where } \lambda_1 = \sqrt{h_1^2-2ah_1+2a^2}$$

so

$$|h_1\rangle = |1\rangle\frac{a}{\lambda_1} + \left(\frac{h_1-a}{\lambda_1}\right)|2\rangle$$

As a check,

$$H|h_1\rangle = a\left(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|\right)\left(|1\rangle\frac{a}{\lambda} + \left(\frac{h_1-a}{\lambda}\right)|2\rangle\right)$$

$$= \frac{a^2}{\lambda_1}|1\rangle + \frac{a^2}{\lambda}|2\rangle - a\left(\frac{h_1-a}{\lambda}\right)|2\rangle\ a\left(\frac{h_1-a}{\lambda}\right)|1\rangle = \left(\frac{h_1a-a^2+a^2}{\lambda}\right)|1\rangle - \left(\frac{h_1a-a^2-a^2}{\lambda}\right)|2\rangle$$

$$= \frac{h_1a}{\lambda_1}|1\rangle + \left(\frac{2a^2-h_1a}{\lambda_1}\right)|2\rangle$$

but h_1 can be determined by

$$\det\begin{bmatrix} a-h_1 & a \\ a & -a-h_1 \end{bmatrix} = 0 = -(a-h_1)(a+h_1) - a^2 = -(-h_1^2 + a^2) - a^2 = h_1^2 - 2a^2 = 0$$

$$\Rightarrow h_1 = \pm\sqrt{2}\,a \quad \Rightarrow \lambda_1 = \sqrt{2a^2 - 2\sqrt{2}a a + 2a^2} = a\sqrt{4-2\sqrt{2}}$$

$$\Rightarrow \frac{a}{\lambda_1} = \frac{1}{\sqrt{4-2\sqrt{2}}} = \frac{1}{\lambda_1'}$$

$$\Rightarrow \frac{h_1 - a}{\lambda} = \frac{\sqrt{2}-1}{\lambda_1'}$$

$$\Rightarrow \frac{2a^2 - h_1 a}{\lambda_1} = \frac{2a^2 - \sqrt{2}a^2}{\lambda_1} = \frac{(2-\sqrt{2})}{\lambda_1'} = \frac{(\sqrt{2}-1)\,h_1}{\lambda_1} = \frac{h_1 - a}{\lambda_1} h_1$$

So

$$H|h_1\rangle = h_1 \frac{a}{\lambda_1}|1\rangle + h_1\left(\frac{h_1-a}{\lambda_1}\right)|2\rangle = h_1\left(\frac{a}{\lambda_1}|1\rangle + \left(\frac{h_1-a}{\lambda_1}\right)|2\rangle\right)$$

$$= h_1 |h_1\rangle !$$

Next, w/ $h_2 = -\sqrt{2}a$, we simply need to make the substitution $h_1 \to h_2$ to find the $|h_2\rangle$ eigenket

$$|h_2\rangle = |1\rangle \frac{a}{\lambda_1} + \left(\frac{h_2 - a}{\lambda_1}\right)|2\rangle \qquad \lambda_1 = \sqrt{h_2^2 - 2ah_2 + 2a^2}$$

$$= |1\rangle \frac{a}{\lambda_1} - a\left(\frac{1+\sqrt{2}}{\lambda_1}\right)|2\rangle \qquad = \sqrt{2a^2 + 2\sqrt{2}a + 2a^2}$$

$$= a\sqrt{4+2\sqrt{2}}$$

as a check

$$H|h_1\rangle = \frac{a^2}{\lambda_1}|1\rangle + \frac{a^2}{\lambda_1}|2\rangle - a^2\left(\frac{1+\sqrt{2}}{\lambda_1}\right)|1\rangle + a^2\left(\frac{1+\sqrt{2}}{\lambda}\right)|2\rangle$$

$$= -\frac{a^2\sqrt{2}}{\lambda_1}|1\rangle + a^2\left(\frac{2+\sqrt{2}}{\lambda_1}\right)|2\rangle = (-\sqrt{2}a)\frac{a}{\lambda_1}|1\rangle - (-\sqrt{2}a)\left(\frac{1+\sqrt{2}}{\lambda_1}\right)|2\rangle$$

$$= h_2\left(\frac{a}{\lambda_1}|1\rangle - \left(\frac{1+\sqrt{2}}{\lambda}\right)|2\rangle\right) = h_2|h_2\rangle$$